Admissible vector fields and quasi-invariant measures[*]

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[*] Appendix to *The Malliavin Calculus, 2nd edition* by D. Bell. Dover Publications, Mineola, New York, 2006.

This appendix, which contains recent work of the author, can be read independently of the preceding material. We use the scheme outlined in Section 5.3 to study the transformation of measure under the flow of a vector field.

Let X denote either a Banach space or a compact smooth finite-dimensional manifold without boundary, equipped with a finite Borel measure γ. Let Z be a C^1 vector field on X and consider the flow $x \mapsto \sigma_s$ generated by Z, defined by the differential equation

$$\dot{\sigma}_s = Z(\sigma_s), \sigma_0 = x.$$

Assume the flow exists for all time s and defines a C^1 function in x. This is automatic in the compact manifold case and holds in the Banach space case provided Z satisfies the global Lipschitz condition

$$||Z(x) - Z(y)|| \leq c||x - y||, \forall x, y$$

for some constant c.

Definition The vector field Z is said to be *admissible* if there exists a random variable Y such that the relation

$$\int_X D\Phi(x) Z d\gamma = \int_X \Phi(x) Y d\gamma$$

holds for all test functions Φ on X. The random variable Y is called the *divergence* of Z and will be denoted by $Div(Z)$ in the sequel.

Example Let $Z \in K$, where K is the space in Section 7.3 and let ξ denote the solution process to the stochastic differential equation (7.22). Then the conclusion of Theorem 7.13 asserts that Z is admissible with respect to the law of ξ_t, for every $t \in (0, 1]$. We note that the argument in Section 7.3 and Theorem 7.13 easily extend to the case where Z is a C^1 map from E into K, with Z and DZ bounded.

Definition We say that γ is *quasi-invariant* (under the flow σ_s) if the measures γ and $\gamma_s \equiv \sigma_s(\gamma)$ are equivalent, for all s.

In a previous paper [A quasi-invariance theorem for measures on Banach spaces, *Trans. Amer. Math. Soc.*, 1985, pp. 290: 851 - 855], the author introduced a method for studying the quasi-invariance of a measure γ defined on a Banach space E, under translation by a vector $h \in E$. The argument entailed studying the quasi-invariance of γ under the family of translations $\{sh, s \in \mathbf{R}\}$, i.e. the flow generated by the (constant) vector field h on E. In this appendix, both the method and the result in Bell[2] will be generalized to the case of non-constant vector fields. The main result is Theorem 3. We begin with the following

Theorem 1 *Suppose γ is quasi-invariant and the family of Radon-Nikodym derivatives $X_s \equiv d\gamma_s/d\gamma$ are differentiable in s. Suppose furthermore the random variables ZX_s are admissible. Then X_s satisfies the differential equation*

$$X_s' = Div(ZX_s). \tag{1}$$

Proof Let Φ be a test function on X. Then

$$\int_X \Phi \circ \sigma d\gamma = \int_X \Phi X_s d\gamma.$$

Replacing Φ by $\Phi \circ \sigma_s^{-1}$ we have

$$\int_X \Phi \circ \sigma_s^{-1} X_s d\gamma = \int_X \Phi d\gamma \tag{2}$$

thus the left hand side is constant in s. Differentiating with respect to s gives

$$0 = \int_X \Big\{ D\Phi(\sigma_s^{-1}(x)) \frac{d}{ds}\sigma_s^{-1}(x) X_s + \Phi \circ \sigma_s^{-1} X_s' \Big\} d\gamma. \tag{3}$$

Differentiating with respect to s in $\sigma_s^{-1}(\sigma_s(x)) = x$, we have

$$\frac{d}{ds}\sigma_s^{-1}(\sigma_s(x)) + D\sigma_s^{-1}(\sigma_s(x))\dot{\sigma}_s(x) = 0$$

thus

$$\frac{d}{ds}\sigma_s^{-1}(\sigma_s(x)) + D\sigma_s^{-1}(\sigma_s(x)) Z(\sigma_s(x)) = 0$$

and we obtain

$$\frac{d}{ds}\sigma_s^{-1}(x) = -D\sigma_s^{-1}(x) Z(x).$$

Substituting this into (3) gives

$$\begin{aligned} 0 &= \int_X \Big\{ - D\Phi(\sigma_s^{-1}(x)) D\sigma_s^{-1}(x) Z(x) X_s + \Phi \circ \sigma_s^{-1} X_s' \Big\} d\gamma \\ &= \int_X \Big\{ - D(\Phi \circ \sigma_s^{-1})(x)(ZX_s) + \Phi \circ \sigma_s^{-1} X_s' \Big\} d\gamma. \end{aligned}$$

Thus

$$\int_X \Phi \circ \sigma_s^{-1} \Big\{ X_s' - Div(ZX_s) \Big\} d\gamma = 0. \tag{4}$$

Since this holds for all test functions Φ, we conclude that (1) holds.

Theorem 2 *Suppose Z is admissible and there exist a family of random variables X_s with $X_0 = 1$ satisfying (1). Then γ is quasi-invariant under the flow generated by Z and*

$$\frac{d\gamma_s}{d\gamma}(x) = X_s(x) = \exp\Big\{\int_0^s Div(Z)(\sigma_{-u}(x))du\Big\}.$$

The proof will require the following

Lemma 1 *Define the pull back $\sigma_s^*(Z)$ of the vector field Z under σ_s by*

$$\sigma_s^*(Z)(x) = D\sigma_s(x)^{-1}Z(\sigma_s(x)). \tag{5}$$

Then $\sigma_s^*(Z) = Z$, i.e. Z is invariant under σ_s.

Proof We have

$$\begin{aligned} 0 &= \frac{d}{ds}\sigma_s^{-1}\big(\sigma_s(x)\big) \\ &= \big(\frac{d}{ds}\sigma_s^{-1}\big)\sigma_s(x) + D\sigma_s^{-1}(\sigma_s(x))\dot{\sigma}_s(x) \\ &= \big(\frac{d}{ds}\sigma_s^{-1}\big)\sigma_s(x) + D\sigma_s^{-1}(\sigma_s(x))Z(\sigma_s(x)). \end{aligned}$$

Substituting this into (5) gives

$$\begin{aligned} \sigma_s^*(Z)(x) &= -\big(\frac{d}{ds}\sigma_s^{-1}\big)\sigma_s(x) \\ &= \big(\frac{d}{ds}\sigma_{-s}\big)(\sigma_s(x) \\ &= Z(x) \end{aligned}$$

as required.

We are now in a position to prove Theorem 2. Assume there exist a family of random variables X_s satisfying $X_0 = 1$ and $X_s' = Div(ZX_s)$, i.e. (4) holds. Reversing the argument used to prove Theorem 1, we deduce from (4) that (2) holds for all test functions Φ. This implies that γ_s is equivalent to γ and

$$\frac{d\gamma_s}{d\gamma} = X_s. \tag{6}$$

Let Φ be an arbitrary test function on X. Using (6) and Lemma 1, we have

$$\int_X \Phi Div(ZX_s)d\gamma = \int_X D\Phi(x)(ZX_s)d\gamma$$

$$= \int_X D\Phi(x)Z.X_s d\gamma = \int_X D\Phi(\sigma_s(x))Z(\sigma_s(x))d\gamma$$
$$= \int_X D(\Phi\circ\sigma_s)(x)\sigma_s^*(Z)d\gamma = \int_X D(\Phi\circ\sigma_s)(x)Zd\gamma$$
$$= \int_X \Phi\circ\sigma_s(x)Div(Z)d\gamma = \int_X \Phi Div(Z)(\sigma_s^{-1}(x))X_s d\gamma.$$

Thus we obtain the key relation

$$Div(ZX_s)(x) = X_s Div(Z)(\sigma_s^{-1}(x)).$$

Substituting this into (1) gives

$$X_s' = X_s Div(Z)(\sigma_{-s}).$$

It follows that

$$X_s = \exp\Big\{\int_0^s Div(Z)(\sigma_{-u})du\Big\}$$

which yields the result.

Definition We say that a function $F : X \mapsto \mathbf{R}$ is *Z-differentiable* at $x \in X$ if $\rho(s) \mapsto F(\sigma_s(x))$ is differentiable at $s = 0$ and denote

$$\rho'(0) = DF(x)Z(x).$$

We give another method for deriving the formula for the Radon-Nikoym derivatives in Theorem 2.

Lemma 2 *Suppose Z is admissible and $F : X \mapsto \mathbf{R}$ is Z- differentiable. Then the vector field ZF is admissible and*

$$Div(ZF) = FDiv(Z) - DF(x)Z.$$

Proof For any test function Φ on X, we have

$$\int_X \Phi F Div(Z)d\gamma = \int_X D(\Phi F)(x)Zd\gamma$$
$$= \int_X \big\{Fd\Phi(x)Z + \Phi DF(x)Z\big\}d\gamma$$
$$= \int_X \big\{d\Phi(x)(FZ) + \Phi DF(x)Z\big\}d\gamma$$
$$= \int_X \Phi\{Div(FZ) + DF(x)Z\}d\gamma.$$

The lemma follows.

Suppose now the hypotheses of Theorem 2 hold and the random variables $X_s(x)$ are Z-differentiable. Applying Lemma 2 to equation (1) gives

$$X_s' = X_s Div(Z) - DX_s(x)Z. \tag{7}$$

Define

$$Y_s(x) = X_s(\sigma_s(x)).$$

Then

$$Y_s' = X_s'(\sigma_s(x)) + DX_s(\sigma_s(x))\dot{\sigma}_s(x)$$

$$= X_s'(\sigma_s(x)) + DX_s(\sigma_s(x))Z(\sigma_s(x))$$

which, using (7)

$$= X_s(\sigma_s(x))Div(Z)(\sigma_s(x))$$

$$= Y_s Div(Z)(\sigma_s(x)).$$

Together with the condition $Y_0 \equiv 1$, this yields

$$Y_s = \exp\Big\{\int_0^s Div(Z)(\sigma_u)du\Big\}.$$

Thus

$$X_s = \exp\Big\{\int_0^s Div(Z)(\sigma_u \circ \sigma^{-1})du\Big\}$$

$$= \exp\Big\{\int_0^s Div(Z)(\sigma_{u-s})du\Big\}$$

$$= \exp\Big\{\int_0^s Div(Z)(\sigma_{-u})du\Big\}.$$

as before.

Theorem 3 *Suppose Z is admissible and there exists $B \subseteq X$ with $\gamma_s(B) = 1$ for all s, such that $Div(Z)$ is defined and Z-differentiable on B. Suppose further that the function $s \mapsto Div(Z)(\sigma_s(x))$ is absolutely continuous for almost all $x \in B$. Then γ is quasi-invariant under Z and*

$$\frac{d\gamma_s}{d\gamma} = \exp\Big\{\int_0^s (DivZ)(\sigma_{-u})du\Big\}.$$

Proof We have

$$\frac{d}{du}(DivZ)(\sigma_{-u}(x)) = D(DivZ)(\sigma_{-u}(x)))\frac{d}{du}\sigma_{-u}(x))$$

$$= -D(DivZ)(\sigma_{-u}(x)))Z(\sigma_{-u}(x)))$$

which, by the invariance of Z (Lemma 1)

$$= -D((DivZ)\circ\sigma_{-u})(x)Z(x).$$

Thus $(DivZ)\circ_{-u}$ is Z-differentiable and

$$D((DivZ)\circ\sigma_{-u})(x)Z(x) = -\frac{d}{du}(Div(Z))(\sigma_{-u}(x))).$$

Integrating with respect to s gives

$$(DivZ)(x) - Div(Z)(\sigma_{-s}(x)) = D\Big[\int_0^s (DivZ)(\sigma_{-u})du\Big](Z(x)). \qquad (8)$$

Define

$$X_s = \exp\Big\{\int_0^s (DivZ)(\sigma_{-u})du\Big\}.$$

By (8) and Lemma 2, ZX_s is admissible and

$$Div(ZX_s)(x) = X_s DivZ - DX_s(x)Z$$

$$= X_s DivZ - X_s\big[DivZ - (DivZ)(\sigma_{-s}(x))\big]$$

$$= X_s(DivZ)(\sigma_{-s}(x))$$

$$= X_s'.$$

The result now follows from Theorem 2.

Example Suppose (i, H, E) is an abstract Wiener space with Gaussian measure γ on E. Let $<.,.>_H$ and $|.|$ denote, respectively, the inner product and norm on H, and define $Z \equiv h$, where h is an element of H. Then

$$\sigma_s(x) = x + sh.$$

It can be shown (cf. eg. [L. Gross, Potential theory on Hilbert space, *J. Funct. Anal.* 1, 1967, pp. 123-181]) that Z is admissible and

$$DivZ(x) =< h, x >$$

where $< h, . >$ denotes a stochastic extension of the linear form$< h., . >_H$ to an L^2 random variable on E, defined by

$$< h, x >= \lim_n < h., P_n x >_H$$

where $\{P_n : E \mapsto H\}$ is any sequence of orthogonal projections converging strongly to the identity on E. Define $B \subseteq E$ to be the set on which the convergence holds. Then it is clear that B is *invariant* under σ_s and for $x \in B$,

$$< h, \sigma_s(x) >=< h, x > +s||h||^2.$$

In particular, $DivZ$ is Z-differentiable. Thus Theorem 3 yields the well-known

Cameron-Martin Theorem *The abstract Wiener measure γ is quasi-invariant under translation by $h \in H$ and, denoting by γ_h the measure $\gamma(. - h)$, we have*

$$\frac{d\gamma_h}{d\gamma} = \exp\big\{ < h, x > -\frac{1}{2}||h||^2\big\}.$$