TOPOLOGICAL PROPERTIES OF FORBIDDING-ENFORCING SYSTEMS

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ABSTRACT
Forbidding-enforcing systems (fe-systems) define classes of languages (fe-families) based on boundary conditions. We characterize f-families having not necessarily finite forbidders and prove that none of the Chomsky families of languages can be defined with an fe-system, hence, fe-systems provide a completely new way of defining classes of languages. We show that f-families map to f-families by morphic maps if and only if the morphism maps symbol to symbol. A morphism mapping e-families to e-families is necessarily surjective.

Keywords: forbidding-enforcing, languages, morphisms, topological spaces

1. Introduction
The rising field of biomolecular computing forced computer scientists to look in the process of computation with a new perspective. New algorithms and computational models using biomolecules have been proposed such as self-assembly, splicing, membrane-computing (see for ex. [6, 7, 12, 13, 16]). These models are based on classical concepts in formal languages such as rewriting techniques and grammar systems and for many can be proven that they have universal computational power. However, the deterministic way of defining languages by “everything that is not allowed is forbidden” employed by these models does not necessarily correspond to the various non-deterministic ways that biomolecules act within one biochemical reaction. Motivated by this non-determinism (considering molecules as strings, and sets of molecules as languages) the authors in [2, 4, 15] use boundary conditions of forbidding and enforcing to introduce another way of defining classes of languages. This new concept is best described with the phrase “everything that is not forbidden is allowed”. In this
sense, the forbidding boundary conditions imply that certain words are not allowed in the language, but any language that does not contain words that are forbidden is allowed. This corresponds to a biochemical condition when certain molecules cannot “survive”. Similarly, the enforcing condition says that in a presence of a set of words, some other words must be present as well, and hence, any language that contains these words is allowed in the family. Again, biochemically, the enforcing conditions provide a model where in certain conditions presence of some molecules imply presence of other (new) molecules. As in any biochemical reaction, the resulting set of molecules, i.e., resulting language can belong to a family of possible sets, all satisfying the initial boundary conditions. Authors in [4, 15] show that solutions of many computational problems such as SAT and Hamiltonian Path Problem can be generated using such forbidding-enforcing systems ($fe$-systems). They also show how these systems can be used to represent duplex DNA molecules and splicing by an enzyme.

From computational point of view, $fe$-systems potentially define large families of languages. In [2] it was shown how this concept can be used to derive languages.

Another way to derive new (families of) languages by $fe$-systems was introduced within the concept of membrane systems [1]. Using membranes, the buildup of the families is in a more restrictive way and hence, new classes of languages can be obtained.

In this paper we concentrate on $fe$-families within the topological (metric) space of formal languages. The metric in the space of formal languages (Definition) is the same as the one implicitly used in [3, 15] and follows a similar approach introduced in symbolic dynamics [10]. This view of formal languages comes to interest only with the introduction of $fe$-systems, since, as it is shown in this paper, none of the Chomsky families of languages correspond to an open or a closed set in the such defined metric space (see subsection 3.2). We observe that this metric characterizes the space of formal languages as Cantor space. $Fe$-systems define families that are closed sets (see [15]), and even more, there is a type of $fe$-systems, i.e., having empty forbidding sets and finite enforcing sets, that define open families of languages (Theorem 20). Hence, $fe$-systems provide a completely new classification of formal languages, different from Chomsky’s hierarchy. The infinite enforcing sets define closed but not open families of languages (Proposition 7). The paper characterizes the open balls in the topology that correspond to $fe$-families. We show that a morphic map is continuous if and only if it is $\lambda$-free (Proposition 14). This naturally corresponds to the characterization of continuous maps on infinite sequences in [14]. We also present several examples of other continuous functions that come from well known operations on languages such as taking products of languages with a fixed language, Kleene star operation, intersection or union with a fixed language etc.

We assume that the reader is familiar with the basic topological notions used in the paper and we refer to a general topology textbook [11]. The paper is organized as follows. Section 2 provides the notation and definitions of $fe$-systems. We consider not-necessarily finite forbidders and we call such $f$-family extended. Theorem 6 provides a characterization of extended $f$-families. Subsection 5 introduces a new concept of “generated” sets. The minimal generated sets generalize in some sense the notion of $E$-extensions introduced in [4]. We use generated and minimal generated sets to
prove that infinite enforcing sets define not open families of languages. The metric on formal languages and the topological properties of $f e$-families are considered in Section 3. Section 4 contains observations about morphisms that map $f e$-families into $f e$-families. We show that morphisms map an $f$-family to an extended $f$-family if and only if the morphism is induced by a symbol-to-symbol map. On the other hand if an $e$-family is mapped into an $e$-family, the morphism is necessarily surjective.

2. Definitions

A finite set of symbols is denoted with $A$ and the set of all words over $A$ which forms a free monoid is denoted by $A^*$. A subset of $A^*$ is called a language. Thus, the power set of $A$, $\mathcal{P}(A^*)$ is the set of all languages consisting of finite words over a finite alphabet $A$. We call it the language set. The length of a word $w \in A$ is denoted with $|w|$ and $A_m$ is the set of all words of length $m$. The empty word, denoted with $\lambda$ has length 0. The set of all words over $A$ with positive length is denoted by $A^+$. On the other hand, an infinite string of symbols from $A$ is called an $\omega$-word and the set of all infinite words over $A$ is denoted by $A^\omega$. So, $A^\omega = \{ \xi_0 \xi_1 \xi_2 \ldots | \xi_i \in A, i \geq 0 \}$.

The word $x \in A^*$ is a factor (subword) of the word $y \in A^*$, denoted $x \text{ sub } y$, if there exist $s, t \in A^*$, such that $y = sx t$. The set of all subwords of a word $x$ is denoted by $\text{ sub } (x)$ and the set of all subwords of a language $K$ by $\text{ sub } (K)$. Clearly, $\text{ sub } (K) = \bigcup_{x \in K} \text{ sub } (x)$. For a language $L$ we denote the set of all words in $L$ with length $\leq m$ with $L^\leq m$.

2.1. Forbidding Systems, $f$-families

We recall the definitions and state a few properties of $f e$-systems. The reader is advised to consult with [4, 15] for more details.

**Definition** A forbidding set $F$ is a (possibly infinite) family of finite nonempty subsets of $A^*$; each element of a forbidding set is called a forbidders.

A forbidding set $F$ is called extended if its forbidders are not necessarily finite.

A language $K$ is said to be consistent with a forbidding $F$, denoted by $K \text{ con } F$, if $F \not\subseteq \text{ sub } (K)$. A language $K$ is consistent with a forbidding set $F$ and we write $F \text{ con } F$, if $K \text{ con } F$ for all $F \in F$. If $K$ is not consistent with $F$ we write $K \not\text{ con } F$.

For a forbidding set $F$ the family of $F$-consistent languages is $\mathcal{L}(F) = \{ K | K \text{ con } F \}$.

The family $\mathcal{L}(F)$ is said to be defined by the forbidding set $F$. We say that a family $\mathcal{L}$ is an $f$-family, if there is a forbidding set $F$ such that $\mathcal{L} = \mathcal{L}(F)$. Two forbidding sets are equivalent if they define the same family of languages. The equivalence relation is denoted by $\sim$. In other words, $F \sim F'$ if and only if $\mathcal{L}(F) = \mathcal{L}(F')$.

**Remark 1** Note that $\mathcal{L}(F) = \mathcal{P}(A^*)$ if and only if $F$ is empty. Also, the empty language $\emptyset$ and $\{ \lambda \}$ are in $\mathcal{L}(F)$ for every $F$.

**Proof.** If the forbidding set is empty, then $F \not\subseteq \text{ sub } (K)$ for all $K \in \mathcal{P}(A^*)$ trivially. Conversely, suppose that $\mathcal{L}(F) = \mathcal{P}(A^*)$. If $F$ is not empty, then there is a forbiddner
and any language $L$ for which $F \subseteq \text{sub}(L)$ is not consistent with $\mathcal{F}$. In particular, $F \ncon \mathcal{F}$. The second observation is trivial. □

The above remark simply says that if nothing is forbidden, then everything is allowed. In what follows we describe some additional properties of $f$-families, in particular, given an $f$-family, we provide a way to construct an extended forbidding set that defines this family.

A forbidding set $\mathcal{F}$ is said to be in minimal normal form if $\mathcal{F}$ is subword free (a word in a forbider cannot be a subword of another word in the same forbider) and subword incomparable (for any two forbidders $F_1$ and $F_2$, $\text{sub}(F_1) \setminus \text{sub}(F_2) \neq \emptyset$ and $\text{sub}(F_2) \setminus \text{sub}(F_1) \neq \emptyset$).

**Theorem 1** [2] For every forbidding set $\mathcal{F}$ there exists a unique equivalent forbidding set $\mathcal{F}'$ in minimal normal form.

Recall that a language $L$ in a family of languages $\mathcal{L}$ is called maximal for $\mathcal{L}$ if for every language $L' \in \mathcal{L}$, $L \subseteq L'$ implies that $L = L'$. The set of maximal languages of an $f$-family $\mathcal{L}(\mathcal{F})$ is denoted by $\mathcal{M}(\mathcal{F})$. In general, if $\mathcal{L}$ is a family of languages, then the set of its maximal languages is denoted by $\mathcal{M}(\mathcal{L})$.

Let $\mathcal{F}$ be a forbidding set and $L \in \mathcal{L}(\mathcal{F})$. Then for every word $w \in L$ we have that $\text{sub}(w) \con \mathcal{F}$. Moreover, we have that $\text{sub}(L) \con \mathcal{F}$, i.e., $\text{sub}(L) \in \mathcal{L}(\mathcal{F})$. Recall that a language is called factorial if it contains all of its factors (subwords). So for a factorial language $L$ we have $\text{sub}(L) = L$. These observations provide the following couple of lemmas.

**Lemma 2** Let $\mathcal{F}$ be a forbidding set and let $L$ be a maximal (with respect to inclusion) language in $\mathcal{L}(\mathcal{F})$. Then:

(i) $L$ is factorial.

(ii) every $L'$ such that $L' \subseteq L$ is in $\mathcal{L}(\mathcal{F})$.

**Proof.** (i) For every language $L$ we have that $L \subseteq \text{sub}(L)$. Since $L \in \mathcal{L}(\mathcal{F})$ we have that $F \nsubseteq \text{sub}(L)$. This implies that $\text{sub}(L) \in \mathcal{L}(\mathcal{F})$. Since $L$ is maximal it follows that $L = \text{sub}(L)$. Hence, $L$ is factorial. (ii) Follows from the fact that every subset of a language consistent with a forbidding set is also consistent with that forbidding set [2, 4, 15]. □

**Remark 2** Note that the above lemma also holds for extended forbidding families.

**Lemma 3** Given two (extended) forbidding sets $\mathcal{F}$ and $\mathcal{F}'$ the following holds:

(i) If $\mathcal{M}(\mathcal{F}) \subseteq \mathcal{L}(\mathcal{F}')$, then $\mathcal{L}(\mathcal{F}) \subseteq \mathcal{L}(\mathcal{F}')$.

(ii) $\mathcal{L}(\mathcal{F}) = \mathcal{L}(\mathcal{F}')$ if and only if $\mathcal{M}(\mathcal{F}) = \mathcal{M}(\mathcal{F}')$.

**Proof.** Follows from the fact that every language in an $f$-family is a subset of a maximal language in that family and every subset of a maximal language of an $f$-family is in that family. □
Lemma 4 Let $\mathcal{F}$ be given in minimal normal form and let $F \in \mathcal{F}$ with $|F| \geq 2$. Then for each $w \in F$ there exists $L_w \in \mathcal{M}(\mathcal{F})$ such that $w \notin L_w$, but $(F \setminus \{w\}) \subset L_w$.

Proof. Let $\mathcal{F}$ and $F$ satisfy the conditions of the lemma. Let $w \in F$. If for all $L \in \mathcal{M}(\mathcal{F})$ $w \in L$, then $L \operatorname{con} F$ implies $L \operatorname{con} F'$ where $F' = F \setminus \{w\}$. So, the forbidding set $\mathcal{F}' = (\mathcal{F} \setminus \{F\}) \cup \{F'\}$ where $F' = F \setminus \{w\}$ would be equivalent to $\mathcal{F}$ which contradicts the minimality of $\mathcal{F}$. Hence, there exists a language $L \in \mathcal{M}(\mathcal{F})$ such that $w \notin L$. Let $\mathcal{L}$ be the set of all such $L$. Suppose that for all $L \in \mathcal{L}$, $(F \setminus \{w\}) \not\subseteq \operatorname{sub}(L)$. Again, this implies that $\mathcal{F} \sim \mathcal{F}'$ contradicting the minimality of $\mathcal{F}$. Hence, there is $L \in \mathcal{L}$ such that $(F \setminus \{w\}) \subseteq L$. □

Let $\mathcal{F}$ be a forbidding set in minimal normal form. With every language $L \in \mathcal{M}(\mathcal{F})$ we associate a tree rooted at $\lambda$. A node $u$ at a tree of $L$ has a child $ua$ for $a \in \mathcal{A}$ if and only if $ua \in L$. Clearly, the trees might be infinite, but each node has at most cardinality of the alphabet number of children. We denote the tree for $L$ with $T_L$.

We construct an equivalent (extended) forbidding set to $\mathcal{F}$ in the following way. An arbitrary symbol from the alphabet set is denoted with $a$.

Let $\mathcal{F}$ be the set of all such $L$. Suppose that for all $L \in \mathcal{L}$, $(F \setminus \{w\}) \not\subseteq \operatorname{sub}(L)$. Again, this implies that $\mathcal{F} \sim \mathcal{F}'$ contradicting the minimality of $\mathcal{F}$. Hence, there is $L \in \mathcal{L}$ such that $(F \setminus \{w\}) \subseteq L$. □

Let $\mathcal{L}$ be the set of all such $L$. Suppose that for all $L \in \mathcal{L}$, $(F \setminus \{w\}) \not\subseteq \operatorname{sub}(L)$. Again, this implies that $\mathcal{F} \sim \mathcal{F}'$ contradicting the minimality of $\mathcal{F}$. Hence, there is $L \in \mathcal{L}$ such that $(F \setminus \{w\}) \subseteq L$. □

We construct an equivalent (extended) forbidding set to $\mathcal{F}$ in the following way. An arbitrary symbol from the alphabet set is denoted with $a$. Let $w = w'a$ be a word such that $w'$ is a node in $T_L$ for some $L \in \mathcal{M}(\mathcal{F})$, but $w$ is not a node in any $T_{L'}$, $L \in \mathcal{M}(\mathcal{F})$. Then $w$ is forbidden by $\mathcal{F}$ in every language of $\mathcal{L}(\mathcal{F})$. We call $w$ strictly forbidden in $\mathcal{L}(\mathcal{F})$. We set $G_w = \{w\}$ and define

$$\mathcal{C} = \{G_w \mid w \text{ is strictly forbidden in } \mathcal{L}(\mathcal{F})\}.$$

We order $\mathcal{C}$ with $G_w \leq G_{w'}$ if $w \in \operatorname{sub}(w')$. Let $\mathcal{G}$ be the set of minimal elements of $\mathcal{C}$ with respect to $\leq$. Then our new extended forbidding set contains $\mathcal{G}$.

Now consider a word $v$ which is a node in $T_L$ but $va$ is not. In addition, $va$ is a node of some other tree $T_{L'}$. Then there must be a node $u$ in $T_{L'}$, such that $ua'$ is not a node in $T_{L'}$ but it is a node in $T_L$, otherwise, $L \subset L'$ and $L$ would not be maximal.

We call words like $va$ and $ua'$ not strictly forbidden for $\mathcal{L}(\mathcal{F})$. Consider

$$\mathcal{P}' = \{va \mid va \text{ is not strictly forbidden for } \mathcal{L}(\mathcal{F})\}.$$

Let $\mathcal{P} = \{w \in \mathcal{P}' \mid \operatorname{sub}(w) \cap \mathcal{P} = \{w\}\}$. For $w = va \in \mathcal{P}$ define

$$H_{va,L} = \begin{cases} \{va\} \cup \{ub \mid ub \in L, \text{ ub is not strictly forbidden for } \mathcal{L}(\mathcal{F}) \} & \text{if } va \notin L \\ \emptyset & \text{if } va \in L \end{cases}$$

Note that if $H_{va,L}$ is not empty then it is not contained in any maximal language in $\mathcal{M}(\mathcal{F})$. We denote:

$$\mathcal{H}_{va} = \cup_{L \in \mathcal{M}(\mathcal{F})} H_{va,L}.$$

Consider the non-empty subsets of $\mathcal{H}_{w}$ that are not contained in any maximal language in $\mathcal{M}(\mathcal{F})$. We order these subsets with $\subseteq$ and let $\mathcal{Q}_{w}$ be the set of minimal subsets of $\mathcal{H}_{w}$ that contain $w$ and that are not contained in any maximal language in $\mathcal{M}(\mathcal{F})$. We include the $\mathcal{Q}_{w}$’s in the new extended forbidding set.
Lemma 5 Let $F$ be given in minimal normal form. Let $\hat{F} = G \cup \{w \in P \mid w \notin Q_w\}$. Then $L(F) = L(\hat{F})$.

Proof. Let $L \in M(F)$. Let $F \in \hat{F}$. By construction of $\hat{F}$, $F$ is either in $G$ or in $Q_w$ for some $w \in P$. If $F \in G$, then $F = \{w\}$ and $w$ is strictly forbidden, such that $w$ is not a node in any maximal language. Hence, $F \not\subseteq L$. If $F \in Q_w$ for some $w \in P$, then $F$ is a minimal subset of $H_w$ that contains $w$ and is not contained in any maximal language. Again, $F \not\subseteq L$. So, $L \in L(\hat{F})$ which establishes $M(F) \subseteq L(\hat{F})$. By Lemma 3, $L(F) \subseteq L(\hat{F})$.

For the converse note that by construction each forbiddner from $F$ is in $\hat{F}$. If $F = \{w\}$ then $F \in G$ since $F$ is in minimal normal form. If $F$ has more than one word, then by Lemma 4 there is a maximal language $L_F \in M(F)$ such that $L_F$ contains all words from $F$ but one. Let $w \in F$ be such that $w \notin L_F$. Since $Q_w$ contains all minimal sets that contain $w$ and are not in any maximal language, we have $F \in Q_w$. Hence, $L(\hat{F}) \subseteq L(F)$. \hfill \qed

The extended forbidding set $\hat{F}$ obtained with the construction above is called maximal set of forbidders. The following example illustrates the construction.

Example 1 Let $A = \{a, b\}$. Consider the following forbidding set $F = \{F_1, F_2\}$ where $F_1 = \{aa, bb\}$ and $F_2 = \{ab, ba\}$. This is an example considered by [4, 2, 15],

as well. There are four maximal languages in $L(F)$: $L_1 = a^*b \cup a^*$, $L_2 = ba^* \cup a^*$, $L_3 = b^*a \cup b^*$, $L_4 = ab^* \cup b^*$. Their trees are depicted in Figure 1. Following the maximal set construction we obtain the following. $C = \{a^ib, b^ia, ab^ia, ba^ib \mid i \geq 1\} \cup \{a^ib, b^ia \mid i > 1\}$. The minimal elements are $G = \{a^i b a \mid i \geq 1\} \cup \{a^i b a, \{bb a\}\}$. Note that $ba$ is a word such that $b$ is a word in $L_1$ but $ba \notin L_1$. Also, $ba \in L_2$. We see that $P = \{aa, bb, ab, ba\}$. We can obtain $H_{ba}$ in the following
way:
\[ H_{ba,L_1} = \{ba, ab, aa, aab, \ldots, a^n b, \ldots \mid n \geq 1\} \]
\[ H_{ba,L_2} = H_{ba,L_3} = \emptyset \]
\[ H_{ba,L_4} = \{ba, bb, ab, abb\} \]

The minimal subsets of \( H_{ba} = H_{ba,L_1} \cup H_{ba,L_4} \) are \( Q_{ba} = \{\{ba, a^n b\mid n \geq 1\} \cup \{\{ba, aa, bb\}, \{ba, abb\}\} \). Using similar arguments we obtain:
\[ Q_{ab} = \{\{ab, b^n a\mid n \geq 1\} \cup \{\{ab, baa\}, \{ab, aa, bb\}\} \]
\[ Q_{aa} = \{\{aa, b^n a\mid n > 1\} \cup \{\{aa, bb\}, \{aa, ab, ba\}\}, \{aa, abb\}\} \]
\[ Q_{bb} = \{\{bb, a^n b\mid n > 1\} \cup \{\{aa, bb\}, \{ab, ba, bb\}\}, \{bb, baa\}\} \]

In this example the extended forbidding set \( \hat{F} \) is a forbidding set, which is not in minimal normal form. The minimal normal form of \( \hat{F} \) is \( F \), and \( F \subseteq \hat{F} \).

From the above construction we have the following characterization of extended \( f \)-families.

**Theorem 6** Let \( \mathcal{L} \) be a family of languages with the set of maximal languages \( \mathcal{M}(\mathcal{L}) \). The following are equivalent:

(i) For all \( L \in \mathcal{M}(\mathcal{L}) \), \( L \) is factorial and if \( L' \subset L \) then \( L' \in \mathcal{L} \).

(ii) \( \mathcal{L} \) is an extended \( f \)-family.

**Proof.** If \( \mathcal{L} \) satisfies (i), then the maximal set of forbidders provides an extended forbidding set \( \hat{F} \) such that \( \mathcal{L} = \mathcal{L}(\hat{F}) \). By Remark 2, we have that the maximal languages of an extended \( f \)-family satisfy (i). \( \square \)

Lemmas 1 and 2 show if \( \mathcal{L} \) is an \( f \)-family, then (i) also holds. However, the next example shows that (i) may define an extended \( f \)-family that is not an \( f \)-family.

**Example 2** Let \( \hat{F} = \{\{aba, ab^2 a, ab^3 a, \ldots\}\} \) and \( \mathcal{L} = \mathcal{L}(\hat{F}) \). Then \( \mathcal{M}(\hat{F}) = \mathcal{M}(\mathcal{L}) \), hence satisfies the two properties of (i), but \( \mathcal{L} \) is not an \( f \)-family.

**Corollary 7** Let \( \mathcal{L} \) be a family of languages with the set of maximal languages \( \mathcal{M}(\mathcal{L}) \). If \( \mathcal{M}(\mathcal{L}) \) is finite and if for all \( L \in \mathcal{M}(\mathcal{L}) \), \( L \) is factorial and \( L' \subset L \) implies \( L' \in \mathcal{L} \), then \( \mathcal{L} \) is an \( f \)-family.

### 2.2. Enforcing Systems, \( e \)-families

**Definition** An enforcing set \( \mathcal{E} \) is a (possibly infinite) family of ordered pairs \( (X, Y) \), where \( X \) and \( Y \) are finite languages and \( Y \neq \emptyset \). Such a pair \( (X, Y) \) is called an enforce.

A language \( K \) satisfies an enforce \( (X, Y) \), denoted \( K \text{ sat } (X, Y) \), if \( X \subseteq K \) implies \( Y \cap K \neq \emptyset \). A language \( K \) satisfies an enforcing set \( \mathcal{E} \), denoted \( K \text{ sat } \mathcal{E} \), if \( K \) satisfies every enforce in \( \mathcal{E} \). If \( K \) does not satisfy \( \mathcal{E} \) we write \( K \text{ nsat } \mathcal{E} \). The family of languages defined by the enforcing set \( \mathcal{E} \) is \( \mathcal{L}(\mathcal{E}) = \{K \mid K \text{ sat } \mathcal{E}\} \). A family of languages \( \mathcal{L} \) is called an \( e \)-family if there is \( \mathcal{E} \) such that \( \mathcal{L} = \mathcal{L}(\mathcal{E}) \).
Observe that every $\mathcal{L}(E)$ contains the language $A^*$. Similarly to forbidding we denote equivalent enforcing sets by $\sim$. Two enforcing sets are equivalent if they define the same family of languages.

In both forbidding and enforcing, we assume that the languages under consideration contain words over a fixed finite alphabet $A$. If this is not specified, then it is assumed that $A$ is the set of all symbols that appear in the words of the set of forbidders and/or in the enforcing set.

The definition for enforcers allows the set $X$ to be empty in the pair $(X, Y)$. In this case, for every language $L$, $L \texttt{ sat}(\emptyset, Y)$ implies that $L \cap Y \neq \emptyset$. Such an enforcer is called brute enforcer [4, 15]. A language $L$ satisfies the enforcer $(X, Y)$ trivially if $X \not\subseteq L$.

Again, we state some trivial observations showing the conditions under which an enforcing set defines the whole language set. From the definition it follows that there is no enforcing set $E$ such that $\mathcal{L}(E) = \emptyset$. (Neither there is a forbidding set $F$ such that $\mathcal{L}(F) = \emptyset$.) If the enforcing set is empty, then the premise “for every enforcer in $E$” is false, hence every language satisfies the enforcing set. An enforcer $(X, Y)$ such that $X \cap Y \neq \emptyset$ is called trivial and such enforcer is satisfied by every language. Hence, we have the following remark.

**Remark 3** $\mathcal{L}(E) = \mathcal{P}(A^*)$ if and only if $E$ is empty or $E$ contains only trivial enforcers.

In what follows, unless otherwise stated all enforcers are non trivial.

**Definition** Given an alphabet $A$, a forbidding-enforcing system (fe-system) is a pair $\Gamma = (F, E)$, where $F$ is a forbidding set over $A$ and $E$ is an enforcing set over $A$. The corresponding forbidding-enforcing family (fe-family) of languages, denoted $\mathcal{L}(F, E)$, consists of all languages that are both consistent with $F$ and satisfy $E$. Hence, $\mathcal{L}(F, E) = \mathcal{L}(F) \cap \mathcal{L}(E)$.

**Remark 4** The family that consists of all languages $\mathcal{P}(A^*)$ is an fe-family $\mathcal{L}(F, E)$ if and only if both $F$ and $E$ are empty.

2.3. Generated Sets

In order to define step by step derivation of languages defined by an enforcing set, the authors in [4] and also in [15] define $E$-extensions. For an enforcing set $E$ and languages $K$ and $L$ we say that $L$ is an $E$-extension of $K$ and write $K \vdash E L$, if for each $(X, Y) \in E$, $X \subseteq K$ implies $L \cap Y \neq \emptyset$. As defined, it is not necessarily the case that $K \subseteq L$, however, in the process of derivation of a language, this premise is included [4, 15]. The authors in [4] take the smallest $E$-extensions steps in order to generate a language that satisfies a given enforcing set $E$. It follows that every language $L$ that contains $X$ from an enforcer $(X, Y) \in E$ has to contain a minimal set defined by the enforcing set $E$. In this paper we are interested in the classes of languages that are defined rather than derived by an fe-system, so we introduce the definition of generated and minimal generated sets. These notions may be seen as
“faster steps” through the $\gamma$-tree defined in [4] and in that sense expand the notion of $\mathcal{E}$-extensions.

Let $\mathcal{E}$ be an enforcing set. Define $\mathcal{E}^{(1)} = \{X \mid (X, Y) \in \mathcal{E}\}$.

**Definition** Let $X \in \mathcal{E}^{(1)}$. A language $g(X)$ generated by $X$ is a set such that the following two conditions hold:

1. $X \subseteq g(X)$
2. $g(X) \text{ sat } (X', Y')$ for every $(X', Y') \in \mathcal{E}$.

A generated language $g_m(X)$ is called minimal if no proper subset of it is a generated set.

Let $\mathcal{E}$ be an enforcing set and let $X \in \mathcal{E}^{(1)}$. We denote the family of generated languages of $X$ with respect to $\mathcal{E}$ with $G_{\mathcal{E}}^X$ or simply $G_X$ when $\mathcal{E}$ is understood. The family of minimal generated languages of $X$ with respect to $\mathcal{E}$ is denoted by $M_{\mathcal{E}}^X$ or simply $M_X$ when $\mathcal{E}$ is understood. Let $M(\mathcal{E}) = \bigcup_{X \in \mathcal{E}^{(1)}} M_X$.

**Remark 5** It follows from the definition of minimal generating sets that if $\mathcal{E}$ is an enforcing set and $X \in \mathcal{E}^{(1)}$, then for every language $L$ such that $L \text{ sat } \mathcal{E}$, $X \subseteq L$ implies that $L$ contains as a subset a set $g_m(X) \in M_X$.

The following example shows how $\mathcal{E}$-extensions and minimal generated sets differ.

**Example 3** Let $\mathcal{E} = \{(\{a, aa\}, \{bb, ba\}), (\{ba\}, \{ab\})\}$ and consider the language $X = \{a, aa\}$ in $\mathcal{E}^{(1)}$. Then $\{a, aa, bb\}$ and $\{a, aa, ba, ab\}$ are minimal generated sets for $X$. The set $\{a, aa, bb, ba\}$ is not a generated set for $X$, but it is an $\mathcal{E}$-extension for the language $X$ and it does not satisfy $\mathcal{E}$. Finally, $\{a, aa, bb, ba, ab\}$ is generated, but it is not minimal.

Lemma 11.14 in [15] shows a redundancy in the enforcing set: if $(X, Y), (X', Y') \in \mathcal{E}$ such that $X \subseteq X'$ and $Y \subseteq Y'$ then $\mathcal{E} \sim \mathcal{E}'$ where $\mathcal{E}' = \mathcal{E} \setminus \{(X', Y')\}$. Although not as simple, the following definition extends the notion of redundancy.

**Definition** Given $\mathcal{E}$, we say that $(X', Y')$ is **redundant** for $\mathcal{E}$, if there exists a $(X, Y) \in \mathcal{E}$, with $X \subseteq X'$ and $Y' \cap g_m(X) \neq \emptyset$ for all $g_m(X) \in M_X^{\mathcal{E}}$ where $\mathcal{E}' = \mathcal{E} \setminus \{(X', Y')\}$.

In particular, if $X \subseteq X'$ and $Y \subseteq Y'$, then $(X', Y')$ is redundant.

**Example 4** Let $\mathcal{E} = \{(\{a\}, \{b\}), (\{b\}, \{c, d\}), (\{a, e\}, \{c, d, f\})\}$, and consider $X = \{a\}$. Observe that any language that satisfies the first two enforcers and contains $a$, has to have either $\{a, b, c\}$ or $\{a, b, d\}$ as subsets, which satisfies the third (redundant) enforcer in both cases.

The following lemma shows that redundant enforcers can be erased from the enforcing set.
Lemma 8 If \((X', Y')\) is redundant for \(\mathcal{E}\), then \(L(\mathcal{E}) = L(\mathcal{E}')\), where \(\mathcal{E}' = \mathcal{E}\setminus\{(X', Y')\}\).

Proof. It is clear that \(L(\mathcal{E}) \subseteq L(\mathcal{E}')\). Let \(L \in L(\mathcal{E}')\). If \(X' \not\subseteq L\), then \(L \in L(\mathcal{E})\). Assume \(X' \subseteq L\) and there is \(X \subseteq X'\) with \(g_m(X) \cap Y' \neq \emptyset\) for all \(g_m(X)\) in \(M_{\mathcal{E}'(X')}\). Since there is at least one \(g_m(X)\) in \(M_{\mathcal{E}'(X')}\) contained in \(L\), \(L \not\subseteq (X', Y')\).

Example 5 Example 3 shows an enforcing set where for each \(X \in \mathcal{E}^{(1)}\), there are finite number of finite languages as minimal generated sets of \(X\). The following examples show that \(M_X\) may contain an infinite language, or it may be an infinite set of infinite languages.

1. Consider \(\mathcal{E} = \{\emptyset, \{\lambda\}\}, \{\{\lambda\}, \{a\}\} \cup \{\{a^i\}, \{a^{i+1}\}\} \mid i \geq 1\}. There is only one minimal generated set in \(M_{\emptyset}\) which is \(a^*\).

2. Let \(\mathcal{E} = \{\{a\}, \{a^2, b^2\}\}, \{\{b\}, \{a^{2i+1}, b^{2i+1}\}\} \mid i > 0\}. Then \(M_{\{a\}}\) is an infinite family of infinite languages. To see this, we construct a tree rooted at \(a\) with two children \(a^2\) and \(b^2\). The enforcer \((\{a^2\}, \{a^4, b^4\})\) defines two children for \(a^2\) to be \(a^4\) and \(b^4\) and the enforcer \((\{b^2\}, \{a^5, b^5\})\) defines two children for \(b^2\) to be \(a^5\) and \(b^5\). We continue in this way for each new node and the corresponding enforcer. Note that the labels of the nodes in the resulting tree are all distinct, the tree is infinite and the union of the labels of an infinite path that starts at the root is a minimal generated set for the set \(X = \{a\}\). Since there are infinite number of such paths, there are infinite number of minimal generated sets and each minimal generated set is infinite.

3. The infinite enforcing set \(\mathcal{E} = \{\{a^i\}, \{a^{2i}\}\} \mid i \geq 1\} contains an infinite number of finite generated sets, i.e., \(g_m(\{a^i\}) = \{a^i, a^{2i}\}\) for \(i \geq 1\).

4. It is obvious that finite enforcing sets have a finite number of finite generated sets.

An enforcing set \(\mathcal{E}\) is said to be finitary if for all \(X \in \mathcal{E}^{(1)}\) there are finite number of enforcers \((X, Y_i)\) in \(\mathcal{E}\). It is shown in [4] that every infinite enforcing set is equivalent to a finitary enforcing set. The following example shows that there exists an infinite finitary enforcing set for which \(M(\mathcal{E})\) is finite.

Example 6 Let \(Z = \{w_1, w_2, \ldots\}\) be an infinite set of words. Consider the enforcing set \(\mathcal{E} = \{\{(w_1, w_2), \{w_3\}\}, \{(w_2, w_3), \{w_4\}\}, \{(w_2, w_4), \{w_1\}\}\} \cup \{(w_n, w_{n+1}), \{w_{n+2}\}\}, \{(w_n, w_{n+2}), \{w_1\}\}, \{(w_1, w_n), \{w_2\}\} \mid n \geq 3\}. It is obvious that this enforcing set is infinite and finitary. Notice that \(M(\mathcal{E})\) is a singleton and its only generated set contains all words in \(Z\). In other words, \(Z = M_X = M_{X'}\) for all \(X, X' \in \mathcal{E}^{(1)}\).

The following lemma shows that an infinite finitary enforcing set with finite \(M(\mathcal{E})\) must have an infinite generated set.

Lemma 9 Let \(\mathcal{E}\) be infinite and finitary, such that \(M(\mathcal{E})\) is finite. Then there exists an infinite generated set.
Proof. Since \( M(\mathcal{E}) \) is finite, we have a finite number of families of minimal generated languages \( M_X \). Denote these families by \( M_1, M_2, \ldots, M_k \), i.e., \( M(\mathcal{E}) = \bigcup_{i=1}^k M_i \). Since there are infinitely many distinct \( X \)'s (due to \( \mathcal{E} \) being infinite) and finitely many \( M_i \)'s, there must exist at least one \( M_j \) such that for infinitely many \( X \)'s in \( \mathcal{E}^{(1)} \), we have \( M_X = M_j \). Let \( g_m(X) \in M_j \). Since \( g_m(X) \) is a generated set for infinitely many \( X \)'s, it follows that \( g_m(X) \) contains all these \( X \)'s as subsets. Hence, \( g_m(X) \) is infinite. (In fact all generated sets in \( M_j \) are infinite.)

We use Lemma 9 in what follows to show that infinite enforcing sets define non-open families of languages.

3. Topological Properties of \( fe \)-families

3.1. Metric in \( \mathcal{P}(A^*) \)

This subsection provides basic topological properties of the space \( \mathcal{P}(A^*) \) using the metric defined in [4] and [15]. This metric comes naturally from the one defined for the \( \omega \)-words in [5] and [14] and the one used in symbolic dynamics (see [9, 10]). Although the metric is natural, the study of the space of all formal languages as a topological (metric) space does not appear in literature. Other topologies on formal languages were considered in [8].

We denote with \( L_1 \triangle L_2 \) the symmetric difference between \( L_1 \) and \( L_2 \).

**Definition (Language Metric)** The distance between any two languages \( L_1 \) and \( L_2 \) in \( \mathcal{P}(A^*) \) is:

\[
d(L_1, L_2) = \begin{cases} 
\frac{1}{2^j}, & \text{for } j = \min\{|w| \mid w \in L_1 \triangle L_2\} \text{ if } L_1 \neq L_2 \\
0, & \text{if } L_1 = L_2
\end{cases}
\]

For example, let \( L_1 = ab^*a \) and \( L_2 = a(bh)^*a \). We have that the shortest word in the symmetric difference of \( L_1 \) and \( L_2 \) is \( aba \) and hence \( d(L_1, L_2) = 2^{-3} \).

It is easy to see that \( d \) defined above is a metric. The open ball centered at \( L \) with radius \( \delta \) is the set of all languages that are at distance less than \( \delta \) from \( L \). It is denoted with \( B_d(L, \delta) \). Clearly, \( K \in B_d(L, \delta) \) if and only if \( K^{<m} = L^{<m} \) for any \( m \) such that \( 2^{-m} < \delta \). There is a close relationship between the language metric and the one defined for \( \omega \)-words in [5]. We recall the following definition.

**Definition (\( \omega \)-word Metric)** The \( \omega \)-word distance between any two words \( \xi \) and \( \eta \) in \( A^\omega \) is:

\[
\rho(\xi, \eta) = \begin{cases} 
\frac{1}{2^j}, & \text{for } j = \min\{i \mid \xi_i \neq \eta_i\} \text{ if } \xi \neq \eta \\
0, & \text{if } \xi = \eta
\end{cases}
\]

As it is well known, \( A^\omega \) equipped with the metric \( \rho \) is homeomorphic to the Cantor space (see for ex. [5, 9, 14]). Open balls in \( A^\omega \) centered at \( \xi \) with radius \( \delta \) is denoted by \( B_\rho(\xi, \delta) \). The following homeomorphism establishes the connection between \( \mathcal{P}(A^*) \) and \( \{0,1\}^\omega \).
Definition Let $K$ be a language. A cylinder set centered at $K$ with bound $m$ is $C(K)_m = \{ L \mid L^{\leq m} = K^{\leq m} \}$.

Note that $K$ and $L$ belong to the same cylinder set with bound $m$ if and only if $L \cap A^{\leq m} = K \cap A^{\leq m}$. The collection of cylinder sets corresponds to the open balls for $\mathcal{P}(A^*)$ and hence is a basis for the topology defined with $d$. Given $m$, $\mathcal{P} = \{ C(K)_m \mid K \in \mathcal{P}(A^*) \}$ forms a finite partition on $\mathcal{P}(A^*)$. For example, if $m = 1$ and $A = \{ a, b \}$, then there are eight cylinder sets in the partition $\mathcal{P}$. Namely, $\mathcal{P} = \{ \emptyset, \{ \lambda \}, \{ a \}, \{ \lambda, a \}, \{ b \}, \{ \lambda, b \}, \{ a, b \}, \{ \lambda, a, b \} \}$. For example, the cylinder set $C(\emptyset)_1$ consists of all languages whose words have length greater or equal to 2. It is easy to see that every language from $\mathcal{P}(A^*)$ belongs to exactly one of these eight cylinders.

The above definition corresponds to the definition for cylinder sets in $X^\omega$ defined with $C_i(a_0 \cdots a_k) = \{ \xi \mid \xi_i \xi_{i+1} \cdots \xi_k = a_0 a_1 \cdots a_k \}$ (see for ex. [9]).

Assume that the symbols in $A$ are ordered and words in $A^*$ are also ordered lexicographically, i.e., there exists an ordering map $\iota : \mathbb{N} \rightarrow A^*$ such that $w(i) = w_i$. Let $X = \{ 0, 1 \}$.

Proposition 10 Let $F_A : \mathcal{P}(A^*) \rightarrow X^\omega$ be a map such that $F_A(L) = \xi$, where $\xi(i) = 1$ if and only if $w_i \in L$. Then $\phi$ is a homeomorphism.

Proof. Obviously, $F_A$ is a bijection. Let $B_d(L, \delta)$ with $\delta > \frac{1}{2m}$ be an open ball in $\mathcal{P}(A^*)$. For each $w \in L^{\leq m}$ let $i(w)$ be the order of $w$ in $A^*$, i.e., $i(w) = \iota^{-1}(w)$. Let $F_A(L) = \xi$ and let $j = \max \{ i(w) \mid w \in L^{\leq m} \}$. Then $F_A(B_d(L, \delta)) = C_0(\xi_0 \cdots \xi_j)$. Hence both $\phi$ and $\phi^{-1}$ are continuous.

Corollary 11 The space $\mathcal{P}(A^*)$ is homeomorphic to the Cantor space.

Proposition 10 shows that $\phi$ maps cylinder sets into cylinder sets. As in $X^\omega$, every cylinder set in $\mathcal{P}(A^*)$ is clopen (closed and open) and hence compact.

3.2. Relationship with the Chomsky Families

Let FIN, REG, CF, CS, RE denote the families of finite, regular, context free, context sensitive and recursively enumerable languages respectively. We see that these families do not correspond to “nice” topological spaces.

A sequence of languages $\{ L_n \}_{n \geq 0}$ is convergent to a language $L$, if for all $m \in \mathbb{N}$ there is $M \in \mathbb{N}$, such that $L_i^{\leq m} = L^{\leq m}$ whenever $i > M$. In this case we use notation $L_n \rightarrow L$.

The next lemma follows directly from the definitions. (See also [3, 15].)

Lemma 12 For each language $L$, the sequence $\{ L_m \}_{m \geq 0} = (L^{\leq m})_{m \geq 0}$ of finite parts of $L$ converges to $L$, i.e., $L^{\leq m} \rightarrow L$.

The above lemma shows that every infinite language is a limit of a sequence of finite languages. Hence the following theorem.
Theorem 13  
(i) Every family of languages $L \neq \mathcal{P}(A^*)$ that contains FIN is not closed.

(ii) None of the classes FIN, REG, CF, CS, RE is topologically closed.

(iii) The FIN family is dense in $\mathcal{P}(A^*)$.

(iv) Every open ball in $\mathcal{P}(A^*)$ contains a non r.e. language.

Proof.  
(i) Follows from the fact that if $K$ is any language and $K \notin L$, then $K^{\leq n} \to K$ and each $K^{\leq n}$ is finite. So, $K$ belongs to the closure of $L$, i.e., $\text{closure}(L) \neq L$.

(ii) Follows from (i)

(iii) Follows from the fact that every language is a limit of a sequence of finite languages Lemma 12.

(iv) Let $R$ be a non r.e. language and define $R^{-j} = R \setminus R^{\leq j}$. For every language $L \in \mathcal{P}(A^*)$ and for every $j > 0$ we have that $d(L, L^{\leq j} \cup R^{-j}) < 2^{-j}$.

The above theorem shows that the well known Chomsky families of languages do not have “nice” properties in this topology. In the study of formal languages these families contain languages that are classified by means much different than topological properties and are separated either by the types of automata that recognize them or by the types of grammars. In this sense, the two regular languages $a^+$ and $a^*$ would be considered very close to each other. But in topological sense, they are at distance 1 from each other (the largest distance possible!). Other topologies on the space of formal languages have also shown to be not suitable for characterizing and classifying the Chomsky hierarchy \cite{8}.

3.3. Continuous Functions

We characterize the homomorphisms that extend to continuous functions on $\mathcal{P}(A^*)$. Note that concatenation of languages $L_1L_2 = \{uv \mid u \in L_1, v \in L_2\}$ makes $\mathcal{P}(A^*)$ a monoid with a zero $\emptyset$ and identity $\{\lambda\}$. This monoid is not finitely generated. A function $\tilde{h} : \mathcal{P}(A^*) \to \mathcal{P}(B^*)$ is an extension of a homomorphism $h : A^* \to B^*$ if $\tilde{h}(L) = \{h(w) \mid w \in L\}$. If $\tilde{h}$ is an extension of $h$ then $\tilde{h}$ is a monoid morphism, i.e., $\tilde{h}(L_1L_2) = \tilde{h}(L_1)\tilde{h}(L_2)$. We say that $h$ is $\lambda$-free, if $h(a) \neq \lambda$ for all $a \in A$.

Proposition 14 Let $\tilde{h} : \mathcal{P}(A^*) \to \mathcal{P}(B^*)$ be an extension of a morphism $h : A^* \to B^*$. Then $\tilde{h}$ is continuous if and only if $h$ is $\lambda$-free.

Proof. Consider a cylinder $C(K)_m$ centered at $K \in \mathcal{P}(B^*)$. Let $L = \{w \mid w$ is in a language in $h^{-1}(K^{\leq m})\}$. We show that the cylinder set $C(L)_m$ centered at $L$ maps into $C(K)_m$. Let $L_1^{\leq m} = L^{\leq m}$ and $h(L_1) = K_1$. Let $u \in K_1$ and $|u| \leq m$. Then there is a word $u' \in L_1$ with $h(u') = u$. Since $L_1^{\leq m} = L^{\leq m}$, we have $u' \in L$ and hence $u = h(u') \in K$, i.e., $K_1^{\leq m} \subseteq K^{\leq m}$. By the symmetry of the argument for $L$ and $L_1$ we have $K_1^{\leq m} = K^{\leq m}$.

Conversely, assume that there is $a \in A$ with $h(a) = \lambda$. Then either $h(A) = \{\lambda\}$ or there is $b$ such that $h(b) \neq \lambda$. Consider $L_0 \subseteq A^*bA^*bA^*$ where $b$ is such that $h(b) \neq \lambda$. 


Let $m > |h(b)|$. For every $n$ define $L_n = L_0 \cup \{a^kb \mid k \geq n\}$. Then $L_0^{\leq n} = L_n^{\leq n}$ for all $n$ but $d(h(L_0), h(L_n)) = 2^{-|h(b)|} > 2^{-m}$. Hence, $h$ is not continuous. In the event that $h(A) = \{\lambda\}$, it is obvious that $h$ is not continuous since $\bar{h}(P(A^*)) = \{\emptyset, \{\lambda\}\}$. 

By Proposition 10, $P(A^*)$ and $\{0,1\}^\omega$ are homeomorphic. Theorem 2.1 in [14] classifies the continuous maps on $X^\omega$. It states that a map $\varphi : X^\omega \to Y^\omega$ is continuous if and only if it is an extension of a totally unbounded (infinite languages map into infinite languages) and sequential (image of a prefix of a word is a prefix of the image of the same word) mapping $\varphi : X^* \to Y^*$. Let $X = Y = \{0,1\}$ and consider two arbitrary alphabets $A$ and $B$. Both $P(A^*)$ as well as $P(B^*)$ are homeomorphic to $X^\omega$. A homomorphism $\bar{h} : P(A^*) \to P(B^*)$ extends to a map $\hat{h}$ such that $\hat{h}F_A = F_B\bar{h}$ where $F_A, F_B$ are the corresponding homeomorphisms defined in Proposition 10 from $P(A^*)$ and $P(B^*)$ to $\{0,1\}^\omega$ respectively (see Figure 2). It is easy to see that any homomorphism $\bar{h} : P(A^*) \to P(B^*)$ defines a sequential $\hat{h}$ and the $\lambda$-free requirement for $h$ corresponds to totally unbounded $\hat{h}$. It follows that Proposition 14 is a corollary of Theorem 2.1 in [14] and Proposition 10.

![Figure 2:](image)

It should be noted, however, that representing $fe$-families as sets of infinite sequences is difficult and unnatural. For example, describing a $f$-family $L(F)$ with $F = \{(ab)\}$ follows our intuition, i.e., every language in this family is a subset of $b^*a^*$.

An attempt to describe the same family using infinite sequences would be quite burdensome since one has to have at hand the order (index) of all words that contain $ab$ as a subword. Then convert the language $K = \{w \mid ab \in \text{sub}(w)\}$, to an infinite sequence $\xi$ as in Proposition 10 and exclude from $X^\omega$ all sequences $\eta$ that contain a $1$ at a position where $\xi$ also contains a $1$.

For the rest of this paper we abstract from using infinite sequences.

**Proposition 15** If $h : A^* \to B^*$ is an injective morphism, then $h$ extends to a continuous $\bar{h}$.

**Proof.** Suppose $h$ does not extend to a continuous $\bar{h}$. Then $h(a) = \lambda$ for some $a \in A$ and $h(\lambda) = \lambda$. Contradiction, since $h$ is one-to-one.

The proof of the following facts are straightforward and are omitted. These are examples of continuous functions that are not morphisms on languages.
Proposition 16 The following functions \( P(A^*) \rightarrow P(A^*) \) are continuous.

1. \( S_L \) defined with \( S_L(K) = KL \) for a fixed \( L \).
2. \( P_L \) defined with \( P_L(K) = LK \) for a fixed \( L \).
3. \( U_L \) defined with \( U_L(K) = L \cup K \) for a fixed \( L \).
4. \( I_L \) defined with \( I_L(K) = L \cap K \) for a fixed \( L \).
5. \( C \) defined with \( C(K) = K^c \) (\( K^c \) is the complement of \( K \)).
6. \( T \) defined with \( T(K) = K^* \).

3.4. Topological Properties of \( fe \)-families

In this section we show that the topology coming from the language metric is a natural framework for investigating \( fe \)-systems. From the properties in the previous section we have the following theorem. It is a direct consequence to the fact that \( fe \)-systems define families of languages that are closed sets ([4, 15]).

Theorem 17 For all \( X, X \in \{ \text{FIN, REG, CF, CS, RE} \} \), there is no \( fe \)-system \( \Gamma \) such that \( L(\Gamma) = X \).

By [4, 15], \( L(F) \) is a closed set for every \( F \). As closed subsets of a compact metric space, \( L(F) \) are compact. The following proposition shows that they are not open.

Proposition 18 Let \( F \) be a nontrivial forbidding set. Then \( L(F) \) is not open.

Proof. Since \( F \) is not empty, there exists a forbider \( F \) in \( F \) such that \( F \) is finite, non empty, and contains words other than the empty word \( \lambda \).

Let \( L \in L(F) \) be given and choose \( a \in A \). Then for every \( s > 0 \) consider \( L' = L \cup \{ a^s w \mid w \in F \} \). Observe that \( L' \in B_d(L, \frac{1}{2s}) \) but \( L' \notin L(F) \), since \( F \subseteq \text{sub}(L') \). Hence, \( L(F) \) is not open. \( \square \)

Note that if \( F \) is trivial, then \( L(F) = P(A^*) \) by Remark 1 and is open by definition. Also, the above proof shows a stronger result, which we state in the following corollary.

Corollary 19 Let \( F \) be given. Then every proper subset \( V \subseteq L(F) \) is not open.

Proof. Proceed as in the previous proof, except that we choose \( L \in V \). Since \( L' \notin L(F) \) then \( L' \notin V \). So, \( V \) is not open. \( \square \)

Authors in [4] and [15] show that \( L(E) \) are closed sets in \( P(A^*) \). Hence these sets are also compact. We now discuss under what conditions they are open.

Proposition 20 If \( E \) is finite, then \( L(E) \) is open.
Proof. If $\mathcal{E}$ is empty, then $\mathcal{L}(\mathcal{E})$ is $\mathcal{P}(A^*)$ and hence is open. Assume $\mathcal{E}$ is not empty and let $\mathcal{E} = \{(X_1, Y_1), \ldots, (X_n, Y_n)\}$. Let $X = X_1 \cup \ldots \cup X_n$ and $Y = Y_1 \cup \ldots \cup Y_n$ and let $m = \max\{|w| \mid w \in X \cup Y\}$. Let $K \in \mathcal{L}(\mathcal{E})$ and let $L$ be any language such that $K \subseteq L \subseteq L^m$. We observe that $L \in \mathcal{L}(\mathcal{E})$. Let $(X_i, Y_i) \in \mathcal{E}$. If $X_i \not\subseteq L$, then $L \not\subseteq (X_i, Y_i)$. If $X_i \subseteq L$, then $X_i \subseteq K$ and since $Y_i \cap L = Y_i \cap K \neq \emptyset$ we have that $L \not\subseteq (X_i, Y_i)$, therefore $L \in \mathcal{L}(\mathcal{E})$. This shows that $B_d(K, \frac{1}{2m}) \subseteq \mathcal{L}(\mathcal{E})$, hence $\mathcal{L}(\mathcal{E})$ is open.

Proposition 20 confirms that the boundary conditions of finite enforcing sets are not very restrictive, and by the observation in Corollary 13 they contain non r.e. languages. In fact, we have the following observation.

Remark 6 For each cylinder set $C = C(K)_m$ the enforcing set $\mathcal{E}_C = \{(\emptyset, \{w\}) \mid w \in K^m\}$ is such that $C \subseteq \mathcal{L}(\mathcal{E}_C)$.

The above proposition shows that an $fe$-system with empty forbidders and finite enforcers is an open set that contains basis elements for the topology on $\mathcal{P}(A^*)$. The finite enforcing sets have potential to provide families of languages that do not contain non r.e. languages. The following observations show that in the case of infinite enforcing sets, the defined family is always non open.

Example 7 If $\mathcal{E}$ is infinite, then $\mathcal{L}(\mathcal{E})$ is not necessarily open. To see this, consider the brute enforcing set $\mathcal{E} = \{(\emptyset, \{w\}) \mid w \in L \text{ and } L \text{ is infinite}\}$. Let $K \in \mathcal{L}(\mathcal{E})$. Since $L$ is infinite, for all $m$ there is $w_m \in L$ such that $|w_m| > m$. Then $L_m = K \setminus \{w_m\}$ is such that $L_m^m = K^m$, but $L_m \not\in \mathcal{L}(\mathcal{E})$.

The following proposition shows that the above example is part of a general rule.

Proposition 21 Let $\mathcal{E}$ be infinite and finitary. Then $\mathcal{L}(\mathcal{E})$ is not open.

Proof. We consider two cases. In the case that $\mathcal{M}(\mathcal{E})$ is infinite we proceed as follows. Consider an infinite sequence of generated sets. Since $\mathcal{L}(\mathcal{E})$ is compact and every generated set is in $\mathcal{L}(\mathcal{E})$, there is a convergent subsequence, say $\{K_n\}_{n \geq 0}$ of generated sets. Denote its limit by $K$. Then, since $\mathcal{L}(\mathcal{E})$ is closed, $K$ is in $\mathcal{L}(\mathcal{E})$. To show that $\mathcal{L}(\mathcal{E})$ is not open we observe that every open ball centered at $K$ contains a language that is not in $\mathcal{L}(\mathcal{E})$. Let $m \geq 0$. There is a $N$ such that for all $n \geq N$, $K_n^m = K^m$. Since there are infinitely many such generated sets $K_n$, there exists at least one (say $K_l, l \geq N$) which contains a word longer than $m$. Remove this word from $K_l$ to obtain $K'_l$. Now $K'_l \not\in \mathcal{L}(\mathcal{E})$ but $d(K'_l, K) < \frac{1}{m}$.

In the case that $\mathcal{M}(\mathcal{E})$ is finite we have from Lemma 9 that there exists an infinite minimal generated set, which we denote by $K$. As in the case of infinite number of generated sets, we show that every open ball centered at $K$ contains a language that is not in $\mathcal{L}(\mathcal{E})$. Since $K$ is infinite, for every $m \geq 0$ there is a word $w_m \in K$, such that $|w_m| > m$. Now for every $m \geq 0$ construct the language $L_m = K \setminus \{w_m\}$. Then for every $m \geq 0$, $L_m \in B_d(K, \frac{1}{2m})$ but $L_m \not\in \mathcal{L}(\mathcal{E})$ because $L_m$ is a proper subset of a minimal generated set.  \[\square\]
Propositions 20 and 21 establish a topological difference between finite and infinite enforcing sets. They show that infinite finitary enforcing sets cannot be equivalent to finite enforcing sets, because the first type of sets describes non-open families of languages and the latter - open. We state this fact in the following corollary.

**Corollary 22** For every infinite finitary enforcing set there is no finite enforcing set equivalent to it and vice versa.

The fact that $L(F,E)$ is always closed follows from the equality $L(F,E) = L(F) \cap L(E)$ and the fact that both $L(F)$ and $L(E)$ are closed. However, this set may not be open. Propositions 18 and 20 give the following.

**Theorem 23** An $fe$-system $\Gamma = (F,E)$ defines an open family of languages if and only if $F = \emptyset$ and $E$ is finite.

Although every cylinder set is included in an $e$-family (Remark 6), there are cylinder sets (open balls) that define families of languages that cannot be defined by $fe$-systems.

**Example 8** For the cylinder set $C = C(\{a, ba\})_2$ there is no $fe$-system $\Gamma$ such that $L(\Gamma) = C$. To see this note that $C$ is open, hence $F = \emptyset$ and $E$ is finite. Then any nonempty combination of words $\{b, aa, ab, bb\}$ must be “excluded” from all languages by means of enforcing only, which is impossible.

The above example extends to the following fact.

**Proposition 24** Let $P$ be a nonempty finite set of words and let $L$ be a family of languages such that $L \in L$ if and only if $P \cap L = \emptyset$. Then for every $fe$-system $\Gamma$, $L(\Gamma) \neq L$.

*Proof.* Let $P$ and $L$ be as defined in the proposition. Suppose there exists an $fe$-system $\Gamma$ such that $L(\Gamma) = L$. Then the language $K = A^* \setminus P$ belongs to $L(\Gamma)$. Let the maximum length of a word in $P$ be $n$. Then all words of length greater than $n$ are in $K$, which implies that $F = \emptyset$. But then the words from $P$ cannot be excluded by enforcing only. Contradiction, hence no such $\Gamma$ exists.

**Corollary 25** Let $C = C(K)_m$ be a cylinder set with $K \neq A^{\leq m}$. Then $C = C(K)_m$ is not an $fe$-family.

*Proof.* Let $P = K^c \cap A^{\leq m}$. Then the corollary follows from Proposition 24.

**Corollary 26** A cylinder set $C = C(K)_m$ is an $fe$-family if and only if $K = A^{\leq m}$.

*Proof.* By letting $F = \emptyset$ and enforcing $A^{\leq m}$ as in Remark 6 we obtain an $fe$-family equal to the cylinder set. The converse follows from Corollary 25.
4. Morphisms and \( f,e \)-systems

In this section we consider morphisms \( h : A^* \to B^* \) that extend to \( \tilde{h} : \mathcal{P}(A^*) \to \mathcal{P}(B^*) \), where \( A \) and \( B \) are two not necessarily distinct alphabets. In the following we write \( h \) instead of \( \tilde{h} \) when it is clear which function is used from the context.

**Proposition 27** Let \( \mathcal{F} \) be a given forbidding set over the alphabet \( A \) and \( h : A^* \to B^* \) be a surjective \( \lambda \)-free morphism. Then \( \mathcal{L}(h(\mathcal{F})) \subseteq h(\mathcal{L}(\mathcal{F})) \).

**Proof.** Let \( L \in \mathcal{L}(h(\mathcal{F})) \) and let \( K = h^{-1}(L) \). Since \( h \) is onto, \( h(K) = L \). Since \( h \) is a morphism, if \( w \in \text{sub}(K) \), then \( h(w) \in \text{sub}(h(K)) \). Let \( F \in \mathcal{F} \), then \( h(F) \in h(\mathcal{F}) \). If \( F \subseteq \text{sub}(K) \), then \( h(F) \subseteq \text{sub}(L) \), which contradicts the fact that \( L \in \mathcal{L}(h(\mathcal{F})) \).

So, \( K \in \mathcal{L}(\mathcal{F}) \) and hence \( L \in h(\mathcal{L}(\mathcal{F})) \). \( \square \)

Here the \( \lambda \)-free requirement is essential, since if \( h(a) = \lambda \) for some \( a \in A \) then the image of the forbidding set \( \{a\} \) is not a forbidding set. Note that an injective morphism is necessarily \( \lambda \)-free which, also, implies that every bijective morphism (isomorphism) is \( \lambda \)-free.

**Proposition 28** Let \( \mathcal{F} \) be given and \( h : A^* \to B^* \) be an injective morphism. Then \( h(\mathcal{L}(\mathcal{F})) \subseteq \mathcal{L}(h(\mathcal{F})) \).

**Proof.** Let \( L \in h(\mathcal{L}(\mathcal{F})) \). Then there is \( K \in \mathcal{L}(\mathcal{F}) \), such that \( h(K) = L \). Let \( F' \in h(\mathcal{F}) \), then there is \( F \in \mathcal{F} \), such that \( h(F) = F' \). Since \( K \in \mathcal{L}(\mathcal{F}) \), \( K \text{ con } F \). This means that there is \( w \in F \), such that \( w \notin \text{sub}(K) \). Consider \( h(w) = w' \). Note that \( w' \in F' \). We show that \( L \text{ con } F' \) by showing that \( w' \notin \text{sub}(L) \). Suppose \( w' \in \text{sub}(L) \). Then there is \( x' \in L \), such that \( w' \in \text{sub}(x') \). Also, there is \( x \in K \), such that \( h(x) = x' \). Since \( h \) is one-to-one, only \( w \) can map to \( w' \), whence \( w \notin \text{sub}(x) \). Contradiction, since \( w \notin \text{sub}(K) \). Therefore, \( L \text{ con } F' \), which implies \( L \in \mathcal{L}(h(\mathcal{F})) \). \( \square \)

The following example shows that that surjectivity is essential in Proposition 27, injectivity is essential in Proposition 28, and equality does not necessarily hold in both Propositions.

**Example 9**

(a) Let \( A = \{a, b, c\} \) and \( B = \{d, e\} \) with \( h(a) = h(b) = d \) and \( h(c) = e \). Let \( \mathcal{F} \) contain only one forbider \( F = \{ac, bc\} \) and let \( K = \{bc\} \). Then \( h(K) = h(F) \) and it is not in \( h(\mathcal{L}(\mathcal{F})) \), but \( h(K) \notin \mathcal{L}(h(\mathcal{F})) \). This example shows that equality does not always hold in Proposition 27. It, also, shows that injectivity is essential in Proposition 28.

(b) To observe that Proposition 27 does not hold if \( h \) is not surjective consider \( A = \{a, b\} \) and \( B = \{c, d, e\} \) with \( h(a) = c \) and \( h(b) = d \). Let \( \mathcal{F} = \{\{aa\}\} \). Then \( L = \{e\} \in \mathcal{L}(h(\mathcal{F})) \), but \( L \notin h(\mathcal{L}(\mathcal{F})) \) since \( h(\mathcal{L}(\mathcal{F})) \subseteq \{c, d\}^+ \). This example, also, shows that equality does not necessarily hold in Proposition 28.
Similar properties hold for enforcing families. In this case the requirement that \( h \) is a morphism is not necessary.

**Proposition 29** Let \( \mathcal{E} \) be given and \( h : A^* \to B^* \) be a surjective map. Then \( \mathcal{L}(h(\mathcal{E})) \subseteq h(\mathcal{L}(\mathcal{E})) \).

**Proof.** Let \( L \in \mathcal{L}(h(\mathcal{E})) \) and let \( K = \{ w \mid h(w) \in L \} \). Since \( h \) is onto, \( h(K) = L \). Suppose \( K \) is sat. \( \mathcal{E} \). Then there is an enforcer \( (X, Y) \in \mathcal{E} \) that is not satisfied by \( K \). The image of this enforcer \( (h(X), h(Y)) \) is in \( h(\mathcal{E}) \) and as such is satisfied by \( L \). Since \( X \subseteq K \), we have that \( h(X) \subseteq L \), and so \( h(K) \cap h(Y) \) is not empty. If \( w' \in h(K) \cap h(Y) \), then \( w' \in h(Y) \) so, there is \( w \in Y \), such that \( h(w) = w' \). The fact that \( w' \in h(K) \) implies that \( w \in K \), since \( K \) contains all preimages of words in \( L \). Consequently, \( w \in K \cap Y \), which means that \( K \cap Y \neq \emptyset \). This contradicts the assumption that \( K \) does not satisfy \( (X, Y) \). Thus, \( K \in \mathcal{L}(\mathcal{E}) \), which implies that \( L \in h(\mathcal{L}(\mathcal{E})) \). \( \square \)

**Proposition 30** Let \( \mathcal{E} \) be an enforcing set and \( h \) be an injective map \( h : A^* \to B^* \). Then \( h(\mathcal{L}(\mathcal{E})) \subseteq \mathcal{L}(h(\mathcal{E})) \).

**Proof.** Let \( L \in h(\mathcal{L}(\mathcal{E})) \). Then there is \( K \in \mathcal{L}(\mathcal{E}) \), such that \( h(K) = L \). Let \( (X', Y') \in h(\mathcal{E}) \) with \( h((X, Y)) = (X', Y') \). If \( X' \not\subseteq L \), then \( L \) satisfies this enforcer trivially. If \( X' \subseteq L \), then \( X \subseteq K \) since \( h \) is injective. Since \( K \) is sat \( (X, Y) \), it follows that \( Y \cap K \neq \emptyset \). Let \( w \in Y \cap K \). Then \( w \in Y \) and \( h(w) \in Y' \). Also, \( w \in K \), which implies that \( h(w) \in L \). This means that \( L \) is sat \( (X', Y') \), i.e., \( L \in \mathcal{L}(h(\mathcal{E})) \). \( \square \)

Consider again \( A = \{ a, b, c \} \) and \( B = \{ d, e \} \) with \( h(a) = h(b) = d \) and \( h(c) = e \). Let \( \mathcal{E} = \{ (ab), \{ cc \} \} \). Since the language \( K = \{ aa \} \) maps into \( L = \{ dd \} \), we have that \( L \in \mathcal{L}(\mathcal{E}) \). However, \( L \not\in \mathcal{L}(h(\mathcal{E})) \). This example shows that equality does not always hold in Proposition 29. It, also, shows that injectivity is essential for Proposition 30. To see that surjectivity is essential for Proposition 29 consider \( A = \{ a, b \} \) and \( B = \{ d, e \} \). Once again, let \( h(a) = c \) and \( h(b) = d \). For \( \mathcal{E} = \{ (a), \{ b \} \} \) and \( L = \{ e \} \) we have that \( L \in \mathcal{L}(h(\mathcal{E})) \) but \( L \not\in \mathcal{L}(h(\mathcal{E})) \). This example, also, shows that equality does not always hold in Proposition 30.

The next corollary follows straightforward.

**Corollary 31** Let \( h : A^* \to B^* \) be a morphism that extends to a morphism \( \tilde{h} : \mathcal{P}(A^*) \to \mathcal{P}(B^*) \). Let \( \mathcal{F} \) be a forbidding set, \( \mathcal{E} \) an enforcing set and \( \Gamma = (\mathcal{F}, \mathcal{E}) \) an fe-system. The following holds:

1. If \( h \) is bijective then \( h(\mathcal{L}(\mathcal{F})) = \mathcal{L}(h(\mathcal{F})) \) and \( h(\mathcal{L}(\mathcal{E})) = \mathcal{L}(h(\mathcal{E})) \).
2. If \( h \) is surjective and \( \lambda \)-free, then \( h(\mathcal{L}(\Gamma)) \subseteq h(\mathcal{L}(\Gamma)) \).
3. If \( h \) is injective, then \( h(\mathcal{L}(\Gamma)) \subseteq \mathcal{L}(h(\Gamma)) \).
4. If \( h \) is bijective, then \( h(\mathcal{L}(\Gamma)) = \mathcal{L}(h(\Gamma)) \).

By the above corollary, a bijective homomorphism, i.e., an isomorphism always maps \( fe \)-families into \( fe \)-families. In the rest of this section we investigate under what
conditions $fe$-families are mapped into $fe$-families. Unlike what the above results may suggest, $f$-families and $e$-families behave differently under the same morphism. Morphisms that increase the length of a word fail to map $f$-families into $f$-families, but may not affect the $e$-families. Similarly, morphisms that are not surjective fail to map $e$-families into $e$-families, but may not affect the $f$-families.

**Proposition 32** Let $h : A^* \to B^*$ be a morphism, such that there is a symbol $a \in A$ with $|h(a)| > 1$. Then there is a forbidding set $F$ such that $h(L(F))$ is not an $f$-family.

**Proof.** Let $a \in A$ with $|h(a)| > 1$. Let $F = \{c \mid c \in A$ and $c \neq a\}$. Then $L(F) = \{L \mid L \subseteq a^*\}$. Consider $h(L(F))$. The language $\{b\} \notin h(L(F))$ for every $b \in B$. Suppose that there is a forbidding set $F'$ such that $h(L(F)) = L(F')$. Then for every $b \in B$, $\{b\}$ must be a forbidden and since the only nonempty subword of $b$ is $b$ itself, $\{b\}$ must be in $F'$. This implies that $L(F') = \{\emptyset, \{\lambda\}\}$, which contradicts the fact that there are non-trivial languages in $h(L(F))$ (for example $h(\{a\}) \in h(L(F))$). Hence the proposition follows. □

We note here that eventhough the image of an $f$-family under such morphisms might not be an $f$-family, it can still be an $fe$-family, as shown in the next example.

**Example 10** Let $A = B = \{a, b\}$ and $h(a) = aa$. Let $F = \{\{b\}\}$. Then $L(F) = \{L \mid L \subseteq a^*\}$. Consider $h(L(F))$. It is not an $f$-family for reasons explained in the above proof, but it is an $fe$-family. Consider the $fe$-system $\Gamma = (F', E')$ where $E' = \{\{a^{2n+1}\}, \{b\} \mid n \geq 1\}$ and $F' = \{\{b\}\}$. Obviously, $h(L(F)) = L(G)$.

The above proposition, also, follows from Lemma 2 since, when $|h(a)| > 1$ for some symbol $a$, we can find an $F$ such that the maximal languages in $M(F)$ map into languages that are not factorial. If there exists a morphism $h : \mathcal{P}(A^*) \to \mathcal{P}(B^*)$ that maps every $f$-family into an $f$-family, then $h(A) \subseteq B$. The following example presents a morphism of this type mapping $f$-families into $f$-families.

**Example 11**

(a) Consider $A = \{a, b\}$ and $B = \{c, d\}$ and a morphism $h$ such that $h(a) = h(b) = c$. If $F = \{\{aa, ab\}, \{ba\}, \{bb\}\}$ then $L(F)$ consists of all languages that are subsets of $a^* \cup b$. Hence, $h(L(F))$ is the family of all languages that are subsets of $c^*$ and $F' = \{\{d\}\}$.

(b) For $h$, $A$, and $B$ as above, set $F = \{\{ab, ba\}, \{aa\}, \{bb\}\}$. Then $L(F)$ consists of languages that don’t have words of length larger or equal to 3. Thus, we can set $F' = \{\{d\}, \{ccc\}\}$.

**Proposition 33** Let $h : A \to B$ extend to a morphism $h : A^* \to B^*$. Then for every $F$ there exists an extended $F'$ such that $h(L(F)) = L(F')$.

**Proof.** Let $F$ be a forbidding. Let $L$ be a maximal language in $h(L(F))$. Then, there is a language $L' \in M(F)$, such that $h(L') = L$. It is sufficient to observe that (i) from Theorem 6 holds for $L$. Let $K \subseteq L$. Then every word in $K$ has a preimage in $L'$. So
there is \( K' \subseteq L' \) such that \( h(K') = K \). Since \( K' \in \mathcal{L}(\mathcal{F}) \) we have that \( K \in h(\mathcal{L}(\mathcal{F})) \).

We now observe that since \( h \) maps symbol to symbol, \( L \) is factorial. Consider \( w \in L \) and \( x \in \text{sub}(w) \). Then there is \( w' \in L' \) and \( x' \in \text{sub}(w) \) such that \( h(w') = w \) and \( h(x') = x \). Since \( L' \) is factorial, it follows that \( x' \in L' \) which implies that \( x \in L \).

The following result states that surjectivity is essential for mapping \( e \)-families into \( e \)-families.

**Proposition 34** Let \( h : A^* \rightarrow B^* \) be a non surjective morphism. Then for every enforcing set \( \mathcal{E} \), \( h(\mathcal{L}(\mathcal{E})) \) is not an \( e \)-family.

**Proof.** The proposition follows from the fact that there exists a word \( w \in B^* \) such that \( h^{-1}(w) = \emptyset \). Suppose there exists \( \mathcal{E}' \) such that \( h(\mathcal{L}(\mathcal{E})) = \mathcal{L}(\mathcal{E}') \). Let \( L \in h(\mathcal{L}(\mathcal{E})) \) and consider a language \( K \in \mathcal{L}(\mathcal{E}') \) that contains \( L \cup \{w\} \) as a subset. (Such a language always exists. In particular, \( B^* \) is one such language.) Then \( K \) is in \( h(\mathcal{L}(\mathcal{E})) \), as well, which contradicts the fact that \( h^{-1}(w) = \emptyset \). Hence, no such \( \mathcal{E}' \) exists.

Although the image of an \( e \)-family under a non surjective morphism is not an \( e \)-family, it could be an \( fe \)-family, as shown in the following example.

**Example 12** Consider \( A = \{a, b\} \), \( B = \{a, b, c\} \) and \( h : A^* \rightarrow B^* \) with \( h(a) = a \) and \( h(b) = b \). Then, given \( \mathcal{E} \) let \( \Gamma = (\{\{c\}\}, \mathcal{E}) \). We have that \( h(\mathcal{L}(\mathcal{E})) = \mathcal{L}(\Gamma) \).

Thus, a morphism that maps \( e \)-families to \( e \)-families is necessarily surjective. The following examples illustrate such morphisms and point to the difficulty in constructing enforcers in the image family. This difficulty stems from the fact that the new enforcers do not necessarily have a preimage in the old enforcers, as seen in the following two examples. In both cases the enforcers defining the image family have completely different structure than the ones we started with.

**Example 13**

(a) Let \( A = \{a, b\} \) and \( B = \{c\} \). Let \( h : A^* \rightarrow B^* \) be a morphism such that \( h(a) = h(b) = c \). Consider

\[
\mathcal{E} = \{(\{aa\}, \{a^3\}) \}, \{(\{aa\}, \{a^5\}) \}, \{(\{ab\}, \{a^5\}) \}, \{(\{ba\}, \{a^5\}) \}, \{(\{bb\}, \{a^5\}) \}.
\]

The minimal generated sets for \( \mathcal{E} \) are: \( g_m(\{aa\}) = \{a^2, a^3, a^4\} \), \( g_m(\{ab\}) = \{ab, a^5\} \), \( g_m(\{ba\}) = \{ba, a^5\} \), and \( g_m(\{bb\}) = \{bb, a^5\} \). They all map into two sets \( \{c^2, c^3, c^4\} \) and \( \{c^2, c^5\} \). Consider a language \( K \in h(\mathcal{L}(\mathcal{E})) \) that contains the word \( cc \). Since \( h^{-1}(cc) = \{aa, ab, ba, bb\} \), \( K \) must contain an image of a minimal generated set, i.e., \( K \) has as a subset either \( \{c^2, c^3, c^4\} \) or \( \{c^2, c^5\} \). Thus, if there is \( \mathcal{E}' \) such that \( h(\mathcal{L}(\mathcal{E})) = \mathcal{L}(\mathcal{E}') \), then there must be an enforcing \( (X, Y) \) in \( \mathcal{E}' \) with \( X = \{cc\} \). Note that \( \{ba, a^5, b^3\} \) and \( \{ba, a^5, b^4\} \) are in \( \mathcal{L}(\mathcal{E}) \) and they map into \( \{c^2, c^3, c^5\} \) and \( \{c^2, c^4, c^5\} \). So we have the following enforcers \( \mathcal{E}' = \{(\{c^2\}, \{c^3, c^5\}) \}, \{(\{c^2\}, \{c^4, c^5\}) \}, \{(\{c^3\}, \{c^4, c^5\}) \} \). We leave it to the reader to check that in this case \( h(\mathcal{L}(\mathcal{E})) = \mathcal{L}(\mathcal{E}') \).
(b) Consider $A = \{a, a', b, b', c, c'\}$ and $B = \{a, b, c\}$. Let $h : A^* \to B^*$ such that $h(a) = h(a') = a$, $h(b) = h(b') = b$, and $h(c) = h(c') = c$. Let

$$E = \{(\{a, b\}, \{abc\}), (\{a', b'\}, \{abc\}), (\{a', c\}, \{abc\}), (\{b', c\}, \{abc\})\}. $$

Then $h(L(E)) = L(E')$ where $E' = \{(a, b, c), \{abc\})$.

Recall that an open map is a function that maps open sets into open sets.

**Corollary 35** Let $h : A^* \to B^*$ be a morphism that defines an open map on the space of languages. If for every $fe$-family $\Gamma = (F, E)$ over alphabet $A$ there is an $fe$-family $\Gamma' = (F', E')$ over $B$ such that $h(L(\Gamma)) = L(\Gamma')$ then $h$ is surjective.

**Proof.** Assume that every $fe$-family maps with $h$ onto an $fe$-family. Then such a family with empty enforcers, or with an empty forbidding set also maps into an $fe$-family. Let $\Gamma = (F, E)$ be such that $F = \emptyset$ and $E$ be finite. Then $L(\Gamma)$ is open and $h(L(\Gamma))$ is open, as well. This means that in $\Gamma' = (F', E')$ the forbidding set must be empty and the set of enforcers must be finite. By Proposition 34 $h$ is surjective.

Unfortunately, the converse does not hold even when $h$ maps symbols to symbols surjectively. Consider the following example.

**Example 14** Let $A = \{a, b\}$, $B = \{c\}$ and $a \mapsto c$, $b \mapsto c$. Let $F = \{{aa, bb}\}, \{ab\}, \{ba\}$ and

$$E = \{(\{aa\}, \{bb\}), (\{bb\}, \{aa\}), (\{a\}, \{a^3\}), (\{b\}, \{b^2\})\}$$

$$\cup \{(\{a^i\}, \{a^{i+1}\}), (\{b^i\}, \{b^{i+1}\}), (\{a^i\}, \{a\}), (\{b^i\}, \{b\}) \mid i \geq 3\}$$

Then $L(\Gamma)$ besides the trivial, contains exactly two languages $\{a, a^3, a^{4}, \ldots\}$ and $\{b, b^3, b^{4}, \ldots\}$. The only non trivial set in $h(L(\Gamma))$ is $K = \{c, c^3, c^{4}, \ldots\}$. What can $\Gamma'$ such that $L(\Gamma') = K$ be? First note that any forbidding set in normal form must be at most a singleton $\{a^i\}$. But in that case, no power of $c$ larger than $i$ is allowed in a language of $L(\Gamma')$. So $F' = \emptyset$. But then, if $K \text{ sat } E'$, we have that $c^* \text{ sat } E'$ too. Hence, it is impossible to exclude $c^2$ using an enforcing set $E'$ only, i.e., there is no $\Gamma'$ such that $L(\Gamma') = h(L(\Gamma))$.

5. Concluding remarks

Forbidding-enforcing systems provide a completely new way to define classes of languages. None of the Chomsky families can be defined in this way. We have investigated topological and morphic properties of $fe$-systems. Although we characterize extended $f$-families, we still do not have a characterization of $f$-families (when all the forbidders are finite). A similar characterization as in Theorem 6 for non-extended $f$-families may be of interest. Also, any characterization of $e$-families remains to be
investigated. We believe that the introduction of minimal generated sets is the first step towards this goal.

Morphisms are natural maps to consider between languages. Although we characterized morphisms that map $f$-families to extended $f$-families and provided results about morphic images of $f$- and $e$-families, the question of what morphisms map $fe$-families into $fe$-families remains open. We conjecture that a morphism maps $e$-families into $e$-families if and only if it is surjective.

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References


