Robust ANOM Tests
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Abstract

This paper present three tests, including two versions of an ANOM permutation test. Type I error rates and power are investigated. The asymmetrical randomization test simulation data will follow. The power study is incomplete.

1 Introduction

A number of different ANOM-type procedures for variances that deal with non-normality have been investigated (see Wludyka and Nelson (1997b, 1999) and Bernard and Wludyka (2001)). These include subsampling, jackknifing, ranking, transformed ranks, and various randomization tests. Somewhat less work has been done on ANOM-type procedures for means when the data are non-normal. Perhaps this is due to the fact that ANOM is actually quite robust to non-normality. The purpose of this paper is to try to quantify exactly how robust ANOM really is to non-normality, and to study some distribution-free alternatives.

For obvious non-normality, when one cannot transform the data to normality, there are a number of possibilities. Bakir (1989) introduced an ANOM procedure based on ranks (ANOMR) for the one-way (possibly unbalanced) layout. In addition, based on the behavior of the many procedures studied for testing variances, transformed ranks and randomization tests also merit study. Therefore, we have compared ANOM, ANOMR, an ANOM-type procedure based on transformed ranks (ANOMTR), and a permutation test version of ANOM (PANOM) for a number of non-normal distributions. We start with a description of each of the three distribution-free techniques, and then compare there behavior using simulation.

2 Distribution-Free ANOM Techniques

2.1 A Rank Test

For his ANOMR procedure Bakir (1989) assumed that the I populations being compared are continuous and differ at most in their location parameters. Each observed value Y_{ij} is replaced by its rank in the combined set of observations (call these R_{ij}'s) and the ANOM is performed on the ranks using critical values provided by Bakir. That is, the $R_{i\bullet}$'s are plotted with decision lines

$$\boxed{\overline{R}_{\bullet\bullet} \pm c(\alpha; I, n).} \tag{1}$$

2.1.1 Critical Values

Bakir (1989) provided exact critical values $c(\alpha; I, n)$ for $I = 3, 4$, various α's, and combinations of small n_i's. For equal sample sizes he suggested a Bonferonni approximation based on the Wilcoxon rank sum statistic, which was found to be satisfactory when α was chosen to coincide with the quantiles of the Wilcoxon rank sum statistic. This leads to α values such as 0.072.

More complete tables of critical values for $\alpha = 0.1, 0.05, 0.01$; $I = 3(1)10$; and $n = 3(1)10$ are provided in Tables 10-12. These values are not the same as the corresponding values in Bakir (1989) for two reasons. First, Bakir (1989) computed his values based on rejecting the null hypothesis of equal means if any sample mean fell on or outside the decision lines. We have chosen to use the more conventional rule of rejecting the null hypothesis only if a sample mean falls outside the decision lines. This makes a difference when one is using ranks since there are only finitely many possible values for the average ranks. This discreteness is also the cause of the second reason for the differences. Bakir (1989) provided critical values to ensure α levels no larger than the nominal. For small values of n this sometimes leads to α levels much smaller than the nominal. In these cases we have chosen to provide critical values with α levels closest to the nominal.

2.1.2 An Asymptotic Procedure

For equal sample sizes and moderate to large samples Bakir (1989) suggested an asymptotic procedure based on the fact that asymptotically the joint distribution of the $|\overline{R}_{i\bullet} - \overline{R}_{\bullet\bullet}|$ is the same as that of the $|\overline{Y}_{i\bullet} - \overline{Y}_{\bullet\bullet}|$ from the usual ANOM (i.e., when the Y_{ij}'s are independent normal with common variance). Thus, one could use the usual ANOM decision limits

$$\boxed{\overline{y}_{\bullet\bullet} \pm h(\alpha; I, N-I)\sqrt{\mathrm{MS_e}}\sqrt{\frac{I-1}{N}}.} \tag{2}$$

Example 2.1 (ANOMR) Three exponential populations with means $\mu_1 = 5, \mu_2 = 1$, and $\mu_3 = 1$ were sampled. The data are in Table 1, and a normal probability plot of the residuals is shown in Figure 1. It is clear from the plot that the data are not normally distributed. The ranks and treatment rank means for these data

Table 1: Data From Three Exponential Populations.

	Population		
	1	2	3
	6.0609	2.0810	1.1794
	17.2851	1.2452	0.0394
	9.3210	1.6899	0.6620
	3.0377	1.7831	0.7759
$\overline{y}_{i\bullet}$	8.93	1.70	0.664

Figure 1: Normal Probability Plot of the Residuals for the Data in Table 1.

are given in Table 2. The overall average rank $\overline{R}_{\bullet\bullet} = (N+1)/2 = 13/2 = 6.5$, and $c(0.01; 3, 4) = 3.75$. The ANOMR chart is shown in Figure 2, and suggest that Population 1 has a significantly high mean and Population 3 has a significantly low mean at $\alpha = 0.01$.

Table 2: Ranks and Treatment Rank Means for the Data in Table 1

	Population		
	1	2	3
	10	8	4
	12	5	1
	11	6	2
	9	7	3
$\overline{R}_{i\bullet}$	10.5	6.5	2.5

Figure 2: ANOMR Chart for the Data in Table 1.

2.2 Transformed Ranks

Transforming the ranks R_{ij} using inverse normal scores leads to a procedure that does not require special critical values. Let

$$E_{ij} \quad = \quad \Phi^{-1}[0.5 + R_{ij}/(2N+1)] \tag{3}$$

where $\Phi^{-1}(x)$ is the inverse of the standard normal distribution function. The usual ANOM procedure is then performed on the E_{ij}'s. More specifically, one computes the $\overline{E}_{i\bullet}$'s and compares then with the decision lines

$$\boxed{\overline{E}_{\bullet\bullet} \pm h(\alpha; I, N-I)\sqrt{\mathrm{MS_e}}\sqrt{\frac{I-1}{N}}} \tag{4}$$

where the $\mathrm{MS_e}$ is computed for the E_{ij}'s.

Example 2.2 (ANOMTR) For the data in Table 1 the ranks and the E_{ij}'s are given in Table 3. For example, y_{12} is the largest value (i.e., has rank $R_{12} = 12$), and (equation (3))

$$E_{12} = \Phi^{-1}(0.5 + 12/25) = \Phi^{-1}(0.98) = 2.0537.$$

Table 3: Ranks, Transformed Ranks, and Summary Statistics for the Data in Table 1

Population					
1		2		3	
Rank	E_{1j}	Rank	E_{2j}	Rank	E_{3j}
10	1.2816	8	0.9154	4	0.4125
12	2.0537	5	0.5244	1	0.1004
11	1.5548	6	0.6433	2	0.2019
9	1.0803	7	0.7722	3	0.3055
$\overline{E}_{i\bullet}$	1.4926		0.7138		0.2551
s_i^2	0.1778		0.0283		0.0180

A normal probability plot of the residuals associated with the E_{ij}'s (Figure 3) suggests the transformation has been successful in establishing normality.

Figure 3: Normal Probability of the Residuals for the E_{ij}'s (Table 3).

From Table 3 one can compute

$$\begin{aligned} \overline{E}_{\bullet\bullet} &= \frac{1.492 + 0.7138 + 0.2551}{3} = 0.820 \\ \mathrm{MS_e} &= \frac{0.1778 + 0.0283 + 0.0180}{3} = 0.0747. \end{aligned}$$

For the ANOM with transformed ranks (ANOMTR) the decision lines (4) are

$$\begin{aligned} 0.820 &\pm h(0.01; 3, 9)\sqrt{0.0747}\sqrt{\frac{2}{12}} \\ &\pm 3.84(0.1116) \\ &\pm 0.429 \\ & (0.391, 1.249). \end{aligned}$$

The $\alpha = 0.001$ decision lines are $(0.208, 1.433)$.

From the ANOMTR chart in Figure 4 one finds that Population 1 has average values that are significantly high at the $\alpha = 0.001$ level, and Population 3 has average values that are significantly low at the $\alpha = 0.01$ level.

Figure 4: ANOMTR Chart with $\alpha = 0.01$ and $\alpha = 0.001$ Decision Lines (Example 2.2).

2.3 A Ramdomization Test

The idea behind a randomization test is to calculate a statistic based on the original samples and then repeatedly rearrange the data (both within and between all the samples) and recalculate the same statistic for each rearrangment. If one assumes the samples come from populations that differ at most in the parameter being estimated by the sample statistic, then the distribution of the rearrangemnents can be used as a reference distribution to measure the unusualness of the original arrangement. Using randomly chosen permutations as the rearragments, one would perform the permutation version of the ANOM (PANOM) as follows. First, compute

$$D_{\max} = \max_i |\overline{y}_{i\bullet} - \overline{y}_{\bullet\bullet}|$$

for the original data arrangement. Then for each of $q = 1, \ldots, \mathcal{N}$ randomly chosen permutations compute

$$D^{(q)}_{\max} = \max_i |\overline{y}_{i(q)\bullet} - \overline{y}_{\bullet\bullet}|.$$

Decision lines for the PANOM chart are

$$\boxed{\overline{y}_{\bullet\bullet} \pm \mathcal{P}_\alpha} \tag{5}$$

where $\mathcal{P}_\alpha$ is the upper α quantile of the randomization reference distribution of the $D^{(q)}_{\max}$.

Example 2.3 (PANOM) For the data in Table 1 critical values are $\mathcal{P}_{0.01} = 4.92$ and $\mathcal{P}_{0.001} = 5.16$ based on 10,000 permutations. The PANOM chart is given in Figure 5 and suggests that Population 1 having a significantly high average at (just barely) the $\alpha = 0.001$ level is the only significant effect.

Figure 5: PANOM Chart with $\alpha = 0.01$ and $\alpha = 0.001$ Decision Lines (Example 2.3).

The second randomization method produces asymmetrical decision decision lines. For each of $q = 1, \ldots, \mathcal{N}$ randomly chosen permutations compute

$$D^{(q)}_{\max} = \max_i(\overline{Y}_{i(q)\bullet} - \overline{Y}_{\bullet\bullet}).$$

and

$$D^{(q)}_{\min} = \min_i(\overline{Y}_{i(q)\bullet} - \overline{Y}_{\bullet\bullet}).$$

Then the upper decision line for the PANOM chart is

$$\boxed{\overline{Y}_{\bullet\bullet} + \mathcal{P}^{+}_{\alpha/2}} \tag{6}$$

where $\mathcal{P}^{+}_{\alpha}$ is the upper $\alpha/2$ quantile of the randomization reference distribution of the $D^{(q)}_{\max}$. The lower decision line for the PANOM chart is

$$\boxed{\overline{Y}_{\bullet\bullet} + \mathcal{P}^{-}_{\alpha/2}} \tag{7}$$

where $\mathcal{P}^{-}_{\alpha}$ is the lower $\alpha/2$ quantile of the randomization reference distribution of the $D^{(q)}_{\min}$.

Example 2.4 (Exponential Data, p. **??**) For the data in Table 1 critical values are $\mathcal{P}^{+}_{0.01} = 4.924$ and $\mathcal{P}^{-}_{0.01} = -3.083$ based on 10,000 permutations. The PANOM chart is given in Figure 6 and suggests that Population 1 has a significantly high average at the $\alpha = 0.02$ level and population 3 has a significantly low average $(0.664 < LDL = 0.682)$.

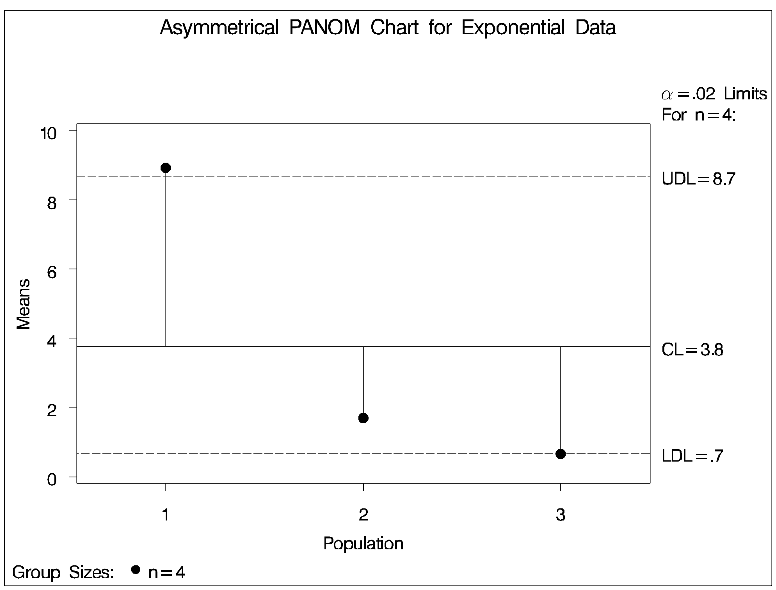

Figure 6: Asymmetrical PANOM Chart with $\alpha = 0.02$ Decision Lines (Example 2.3).

2.4 Comparison of the Three Procedures

For the exponential data in Example 2.1 the three analyses (Examples 2.1, 2.2, and 2.3) all give slightly different results. In order to better compare these procedures we studied their behavior using simulation for a variety of underlying distributions. The eight distributions studied are listed in Table 4. Tables 6 - 9 are the simulated significance levels for the four procedures with the different underlying distributions and various I and n combinations. These results are summarized in Table 5, where the

Table 4: Distributions Studied

Name	Distribution	Details
Normal	$N(0,1)$	
$t(3)$	$t(3)$	
Exp	Exponential	$f(x) = \mathrm{e}^{-x}$
Dexp	Double Exponential	$f(x) = \mathrm{e}^{-\|x\|}$
Kur(6)	Kurtosis 6	$0.66268 + 0.10189Z^3$
Sk(1.75)	Skewness 1.75	$0.3995 + 0.9297Z + 0.3995Z^2 - 0.03647Z^3$
Kur(75)	Kurtosis 75	$-0.05134 - 2.91756Z + 0.05134Z^2 + 0.87133Z^3$
Mix	Normal Mixture	$0.7N(0,1) + 0.3N(5,9)$

variances of the simulated values about the nominal level are given for each procedure.

Table 5: Sums of Squares for Differences From the Nominal Level of Significance ($\alpha = 0.05$)

n	ANOM	ANOMR	ANOMRTR	PANOM
3	0.015764	0.007398	0.007916	0.002443
4	0.015607	0.001454	0.002799	0.001750
5	0.014242	0.001107	0.001773	0.001935
6	0.013667	0.000707	0.001154	0.001618
7	0.013699	0.000561	0.000956	0.001586
8	0.012762	0.000559	0.000864	0.001840
9	0.011188	0.000433	0.000646	0.001709
10	0.010867	0.000372	0.000490	0.001534
sum	0.1078	0.0126	0.0166	0.0144

Table 6: Simulated Significance Levels for I Treatments Each with $n = 3$ Replicates and $\alpha = 0.1$

I		Normal	$t(3)$	Exp	Dexp	Kur(6)	Sk(1.75)	Kur(75)	Mix
3	ANOM	0.1029	0.0808	0.0775	0.0865	0.0831	0.0834	0.0974	0.0984
	ANOMR	0.1312	0.1250	0.1287	0.1287	0.1313	0.1275	0.1261	0.1257
	ANOMRTR	0.1045	0.1002	0.1005	0.1045	0.1064	0.1025	0.1022	0.1025
	PANOM	0.0940	0.0920	0.0760	0.0840	0.0950	0.0940	0.0905	0.0875
4	ANOM	0.0917	0.0864	0.0880	0.0890	0.0877	0.0928	0.0905	0.1013
	ANOMR	0.0635	0.0684	0.0695	0.0696	0.0677	0.0690	0.0668	0.0700
	ANOMRTR	0.0862	0.0945	0.0919	0.0917	0.0915	0.0944	0.0884	0.0944
	PANOM	0.0775	0.0995	0.0850	0.0855	0.0825	0.0895	0.0870	0.1030
5	ANOM	0.1025	0.0940	0.1066	0.0957	0.0954	0.1026	0.0954	0.0978
	ANOMR	0.0908	0.0888	0.0905	0.0854	0.0868	0.0901	0.0856	0.0844
	ANOMRTR	0.0975	0.0953	0.0981	0.0896	0.0968	0.0961	0.0954	0.0941
	PANOM	0.0825	0.0915	0.1055	0.0845	0.0985	0.0970	0.0975	0.1075
6	ANOM	0.1030	0.1093	0.1138	0.1050	0.1083	0.1103	0.1010	0.1043
	ANOMR	0.1016	0.0966	0.1031	0.1013	0.0955	0.0971	0.1038	0.1008
	ANOMRTR	0.0968	0.0971	0.1015	0.0990	0.0960	0.1004	0.0981	0.0963
	PANOM	0.1020	0.1000	0.0950	0.1030	0.0945	0.1020	0.1060	0.0985
7	ANOM	0.1039	0.1160	0.1243	0.1159	0.1152	0.1196	0.1082	0.1019
	ANOMR	0.1122	0.1133	0.1137	0.1084	0.1097	0.1104	0.1158	0.1085
	ANOMRTR	0.1019	0.1091	0.1055	0.1040	0.0996	0.1004	0.1040	0.1014
	PANOM	0.1045	0.0995	0.0985	0.1050	0.0945	0.0940	0.1105	0.0980
8	ANOM	0.0956	0.1235	0.1421	0.1183	0.1207	0.1303	0.1271	0.1029
	ANOMR	0.0804	0.0808	0.0897	0.0835	0.0813	0.0818	0.0911	0.0814
	ANOMRTR	0.1029	0.1116	0.1117	0.1091	0.1079	0.1053	0.1151	0.1046
	PANOM	0.0965	0.0990	0.1140	0.0925	0.0965	0.0990	0.0990	0.0975
9	ANOM	0.1032	0.1485	0.1390	0.1199	0.1336	0.1469	0.1387	0.1012
	ANOMR	0.1070	0.1013	0.1015	0.1040	0.1043	0.1063	0.1058	0.1046
	ANOMRTR	0.1111	0.1087	0.1089	0.1086	0.1075	0.1119	0.1118	0.1089
	PANOM	0.1025	0.1000	0.1080	0.0995	0.0925	0.1040	0.1100	0.1085
10	ANOM	0.0982	0.1538	0.1460	0.1301	0.1362	0.1397	0.1423	0.0941
	ANOMR	0.1087	0.1120	0.1030	0.1101	0.1091	0.1087	0.1152	0.1049
	ANOMRTR	0.1139	0.1146	0.1078	0.1138	0.1101	0.1095	0.1123	0.1109
	PANOM	0.0990	0.1080	0.0865	0.1005	0.0895	0.0885	0.0995	0.1005

Table 7: Simulated Significance Levels for I Treatments Each with $n = 4$ Replicates and $\alpha = 0.1$

I		$t(3)$	Normal	Exp	Dexp	Kur(6)	Sk(1.75)	Kur(75)	Mix
3	ANOM	0.0972	0.0868	0.0865	0.0924	0.0865	0.0914	0.0928	0.1017
	ANOMR	0.0724	0.0776	0.0779	0.0785	0.0773	0.0796	0.0763	0.0801
	ANOMRTR	0.0956	0.0963	0.0970	0.0932	0.0940	0.0991	0.0905	0.0950
	PANOM	0.0900	0.1045	0.0970	0.1010	0.0965	0.1140	0.0930	0.1130
4	ANOM	0.1007	0.0876	0.0885	0.0926	0.0932	0.0944	0.1011	0.0974
	ANOMR	0.1083	0.1077	0.1050	0.1095	0.1096	0.1127	0.1145	0.1080
	ANOMRTR	0.0952	0.0912	0.0903	0.0946	0.0942	0.0977	0.0959	0.0931
	PANOM	0.0850	0.1050	0.1000	0.1010	0.0945	0.1160	0.0975	0.1075
5	ANOM	0.1038	0.0997	0.1114	0.1005	0.0970	0.0998	0.1017	0.1020
	ANOMR	0.1080	0.0988	0.1037	0.1020	0.1047	0.1053	0.1031	0.1051
	ANOMRTR	0.1004	0.0954	0.1045	0.0946	0.1002	0.0945	0.0958	0.1008
	PANOM	0.1055	0.1115	0.1000	0.1055	0.0940	0.1095	0.0940	0.1000
6	ANOM	0.1032	0.1086	0.1122	0.1085	0.1045	0.1167	0.1052	0.1024
	ANOMR	0.1034	0.1011	0.1040	0.1010	0.1015	0.1059	0.1023	0.1036
	ANOMRTR	0.0971	0.1012	0.0978	0.0947	0.0970	0.1009	0.0979	0.0972
	PANOM	0.0890	0.0980	0.0990	0.0970	0.1070	0.1140	0.0955	0.1025
7	ANOM	0.0984	0.1209	0.1162	0.1076	0.1166	0.1152	0.1148	0.0986
	ANOMR	0.0994	0.1012	0.1021	0.1001	0.1008	0.1034	0.0941	0.1019
	ANOMRTR	0.1008	0.0975	0.0955	0.0993	0.1008	0.0979	0.0996	0.0982
	PANOM	0.0880	0.1065	0.1030	0.1090	0.1060	0.1020	0.1010	0.0890
8	ANOM	0.0971	0.1301	0.1282	0.1130	0.1198	0.1254	0.1356	0.0986
	ANOMR	0.1010	0.1005	0.1021	0.1042	0.0998	0.1075	0.1060	0.1029
	ANOMRTR	0.0994	0.0999	0.1006	0.1046	0.1009	0.1049	0.1031	0.0987
	PANOM	0.1015	0.0940	0.1000	0.0980	0.1010	0.0950	0.1020	0.1010
9	ANOM	0.0985	0.1444	0.1354	0.1232	0.1297	0.1350	0.1490	0.1025
	ANOMR	0.1039	0.1017	0.1016	0.1044	0.1067	0.1103	0.1044	0.1044
	ANOMRTR	0.1041	0.1046	0.1018	0.1035	0.1051	0.1081	0.1012	0.1077
	PANOM	0.1030	0.0940	0.0960	0.1030	0.1105	0.0975	0.0900	0.1105
10	ANOM	0.0978	0.1568	0.1494	0.1298	0.1357	0.1404	0.1723	0.1029
	ANOMR	0.1097	0.1032	0.1121	0.1113	0.1083	0.1098	0.1100	0.1061
	ANOMRTR	0.1079	0.1039	0.1119	0.1110	0.1089	0.1075	0.1060	0.1092
	PANOM	0.1060	0.1125	0.1085	0.0950	0.0970	0.1030	0.1020	0.1055

Table 8: Simulated Significance Levels for I Treatments Each with $n = 5$ Replicates and $\alpha = 0.1$

I		$t(3)$	Normal	Exp	Dexp	Kur(6)	Sk(1.75)	Kur(75)	Mix
3	ANOM	0.0980	0.0876	0.0877	0.0873	0.0851	0.0854	0.0940	0.1005
	ANOMR	0.1054	0.1080	0.1036	0.1011	0.1030	0.1040	0.1073	0.1082
	ANOMRTR	0.0928	0.1010	0.0979	0.0939	0.0935	0.0944	0.1000	0.1038
	PANOM	0.1035	0.1050	0.0950	0.0850	0.0880	0.0920	0.1105	0.0980
4	ANOM	0.0989	0.0941	0.0930	0.0984	0.0981	0.0951	0.0983	0.1017
	ANOMR	0.0944	0.0957	0.0885	0.0976	0.0963	0.0977	0.0996	0.0972
	ANOMRTR	0.0961	0.0969	0.0940	0.1021	0.0996	0.1045	0.0963	0.0998
	PANOM	0.1035	0.1080	0.0900	0.1020	0.1130	0.0945	0.1065	0.1015
5	ANOM	0.0944	0.0993	0.0983	0.1008	0.1024	0.1046	0.0988	0.1009
	ANOMR	0.0969	0.0967	0.1008	0.0976	0.1005	0.1021	0.0921	0.0970
	ANOMRTR	0.0942	0.0958	0.0968	0.0926	0.1004	0.1011	0.0888	0.0960
	PANOM	0.0895	0.1020	0.0935	0.1075	0.1030	0.1090	0.0940	0.1155
6	ANOM	0.0983	0.1058	0.1052	0.1054	0.1108	0.1097	0.1100	0.1007
	ANOMR	0.0882	0.0919	0.0890	0.0881	0.0896	0.0970	0.0869	0.0909
	ANOMRTR	0.0978	0.0970	0.0952	0.0945	0.0983	0.1013	0.0980	0.0976
	PANOM	0.0970	0.1015	0.1000	0.0925	0.1105	0.1035	0.1055	0.1045
7	ANOM	0.0999	0.1230	0.1226	0.1109	0.1127	0.1131	0.1232	0.1029
	ANOMR	0.0919	0.0907	0.0919	0.0946	0.0957	0.0925	0.0926	0.0947
	ANOMRTR	0.0969	0.0982	0.0999	0.0983	0.0989	0.0952	0.0967	0.0995
	PANOM	0.1075	0.0895	0.1150	0.1070	0.0955	0.1040	0.1000	0.0930
8	ANOM	0.1063	0.1349	0.1231	0.1127	0.1182	0.1225	0.1363	0.1023
	ANOMR	0.1076	0.0994	0.0992	0.1012	0.0986	0.1055	0.1006	0.1020
	ANOMRTR	0.1073	0.1106	0.0961	0.0964	0.0988	0.1026	0.0962	0.1019
	PANOM	0.1130	0.1120	0.0995	0.0940	0.0985	0.1060	0.1110	0.1090
9	ANOM	0.0975	0.1394	0.1351	0.1190	0.1296	0.1328	0.1551	0.1031
	ANOMR	0.0979	0.0989	0.0968	0.0982	0.0988	0.1045	0.0958	0.0964
	ANOMRTR	0.1003	0.1030	0.1033	0.0969	0.1049	0.1051	0.0963	0.1075
	PANOM	0.0960	0.0960	0.0955	0.1015	0.1085	0.1015	0.1050	0.0970
10	ANOM	0.0936	0.1531	0.1437	0.1211	0.1312	0.1405	0.1709	0.0965
	ANOMR	0.1056	0.1063	0.1078	0.1083	0.1114	0.1074	0.1135	0.1053
	ANOMRTR	0.0992	0.1064	0.1003	0.1052	0.1022	0.1101	0.1091	0.1049
	PANOM	0.0905	0.0925	0.0960	0.0995	0.0950	0.1100	0.1000	0.1040

Table 9: Simulated Significance Levels for I Treatments Each with $n = 10$ Replicates and $\alpha = 0.1$

I		$t(3)$	Normal	Exp	Dexp	Kur(6)	Sk(1.75)	Kur(75)	Mix
3	ANOM	0.1026	0.0851	0.0927	0.0988	0.0990	0.1014	0.0916	0.0976
	ANOMR	0.1085	0.1034	0.1053	0.1038	0.1056	0.1078	0.1043	0.0994
	ANOMRTR	0.1019	0.0924	0.1010	0.1000	0.0999	0.1011	0.1007	0.0976
	PANOM	0.1015	0.0975	0.0985	0.1080	0.1050	0.1110	0.1070	0.1060
4	ANOM	0.1014	0.0931	0.0968	0.0973	0.1044	0.0919	0.0968	0.0989
	ANOMR	0.0986	0.0990	0.1063	0.0993	0.1003	0.1026	0.1025	0.1017
	ANOMRTR	0.0974	0.0991	0.0986	0.0958	0.1004	0.0948	0.0992	0.0993
	PANOM	0.1025	0.1000	0.1025	0.1060	0.1070	0.0880	0.1020	0.0975
5	ANOM	0.0995	0.1049	0.0966	0.1026	0.1059	0.1015	0.1014	0.0964
	ANOMR	0.0978	0.1026	0.0998	0.0991	0.1035	0.1008	0.1017	0.0970
	ANOMRTR	0.0986	0.1013	0.0964	0.0990	0.0976	0.1015	0.0980	0.0990
	PANOM	0.0995	0.1060	0.0970	0.1000	0.1060	0.1020	0.1015	0.0990
6	ANOM	0.1009	0.1087	0.1006	0.1023	0.1050	0.0994	0.1131	0.1030
	ANOMR	0.1056	0.1008	0.0994	0.1007	0.1038	0.1028	0.0989	0.1070
	ANOMRTR	0.0987	0.0970	0.0968	0.0897	0.0975	0.0972	0.0984	0.1002
	PANOM	0.1000	0.0965	0.1110	0.0930	0.0890	0.0945	0.0995	0.0990
7	ANOM	0.0985	0.1203	0.1109	0.1042	0.1058	0.1027	0.1315	0.1003
	ANOMR	0.0988	0.1067	0.1035	0.0986	0.0977	0.0995	0.0993	0.1006
	ANOMRTR	0.0963	0.1052	0.1025	0.0952	0.0962	0.0969	0.0983	0.0976
	PANOM	0.0940	0.0970	0.1105	0.1005	0.0850	0.0955	0.1030	0.1060
8	ANOM	0.1002	0.1228	0.1113	0.1057	0.1093	0.1094	0.1515	0.0989
	ANOMR	0.0943	0.0971	0.1011	0.0921	0.0941	0.0964	0.0986	0.0953
	ANOMRTR	0.1007	0.0960	0.0996	0.0942	0.0984	0.0964	0.0980	0.0984
	PANOM	0.1065	0.0845	0.1025	0.1020	0.1020	0.1000	0.0980	0.0955
9	ANOM	0.1052	0.1339	0.1107	0.1051	0.1160	0.1151	0.1600	0.1016
	ANOMR	0.0991	0.0963	0.0926	0.0959	0.0962	0.1022	0.0980	0.1015
	ANOMRTR	0.1048	0.0948	0.0968	0.0972	0.1004	0.1032	0.1040	0.0986
	PANOM	0.1095	0.0905	0.0890	0.0980	0.0880	0.1090	0.1015	0.1005
10	ANOM	0.1046	0.1483	0.1280	0.1120	0.1206	0.1168	0.1804	0.1036
	ANOMR	0.1006	0.1002	0.1037	0.0996	0.0971	0.0998	0.1020	0.1057
	ANOMRTR	0.1020	0.0989	0.1073	0.0973	0.1049	0.1004	0.1067	0.1039
	PANOM	0.1040	0.0985	0.0985	0.0920	0.0940	0.0880	0.1060	0.0920

3 Appendix - Critical Values for ANOMR

Table 10: Critial Values $c(\alpha; I, n)$ for ANOMR

Level of Significance = 0.1

	Number of Means Being Compared, I							
n	3	4	5	6	7	8	9	10
3	2.34	3.84	5.00	6.17	7.34	8.84	10.00	11.17
4	3.00	4.25	5.75	7.25	8.75	10.25	11.75	13.25
5	3.20	4.90	6.60	8.30	10.00	11.70	13.40	15.10
6	3.50	5.34	7.17	9.00	10.84	12.84	14.68	16.68
7	3.86	5.79	7.72	9.79	11.72	13.94	16.00	18.08
8	4.13	6.13	8.38	10.51	12.63	14.88	17.13	19.38
9	4.34	6.62	8.78	11.07	13.46	15.84	18.24	20.62
10	4.60	6.90	9.30	11.70	14.20	16.80	19.30	21.80

Table 11: Critial Values $c(\alpha; I, n)$ for ANOMR

Level of Significance = 0.05

	Number of Means Being Compared, I							
n	3	4	5	6	7	8	9	10
3	2.67	4.17	5.34	6.50	8.00	9.17	10.67	11.84
4	3.25	4.75	6.25	7.75	9.50	11.00	12.75	14.25
5	3.60	5.50	7.20	8.90	10.80	12.70	14.40	16.30
6	4.00	6.01	7.84	9.84	11.84	14.00	16.01	18.01
7	4.30	6.36	8.58	10.79	12.86	15.08	17.30	19.65
8	4.63	6.88	9.26	11.51	13.88	16.25	18.63	21.00
9	4.89	7.28	9.79	12.17	14.78	17.28	19.79	22.40
10	5.30	7.80	10.30	13.10	15.60	18.30	21.00	23.70

Table 12: Critial Values $c(\alpha; I, n)$ for ANOMR

Level of Significance = 0.01

	Number of Means Being Compared, I							
n	3	4	5	6	7	8	9	10
3	3.00	4.50	6.00	7.17	8.67	10.17	11.34	12.84
4	3.75	5.50	7.25	9.00	10.75	12.50	14.00	16.00
5	4.40	6.30	8.20	10.30	12.40	14.30	16.40	18.50
6	4.84	7.01	9.34	11.34	13.84	16.01	18.17	20.50
7	5.29	7.65	10.01	12.51	15.01	17.51	19.87	22.37
8	5.63	8.25	10.75	13.38	16.13	18.88	21.25	24.00
9	6.00	8.73	11.57	14.40	16.90	20.07	21.34	24.18
10	6.40	9.30	12.20	15.20	17.50	21.20	19.60	22.80

For values of $10 < n < 40$ the equation

$$\ln(c) = c_1 - \frac{c_2}{\sqrt{I}} - \frac{c_3}{\sqrt{n}} \tag{8}$$

and the constants in Table 13 produce critical values accurate to three significant figures. For values of $n \geq 40$ the asymptotic critical values

$$h(\alpha; I, \infty)\sqrt{\frac{(N+1)(I-1)}{12}}$$

are correct to three significant figures.

Table 13: Constants for Use in Equation (8) to Computing Critical Values $c(\alpha; I, n)$ for $10 < n < 40$

α	c_1	c_2	c_3
0.1	1.2132	−0.8271	−0.09709
0.05	1.2562	−0.6601	−0.13612
0.01	1.3698	−0.4522	−0.2504