Poissons remarkable calculation - a trick or a method?

Denis Bell

University of North Florida

Integrating the Gaussian function

The integral

$$I \equiv \int_{-\infty}^{\infty} e^{-x^2}\, dx$$

plays a fundamental role in probability and statistics.

It is well-known that the integrand does not have an elementary antiderivative. However the integral can be evaluated by the following remarkable trick attributed to Poisson.

Squaring I, we have

$$I^2 = \Big(\int_{-\infty}^{\infty} e^{-x^2}\, dx\Big)\Big(\int_{-\infty}^{\infty} e^{-y^2}\, dy\Big)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dxdy$$

Transforming to polar coordinates

$$x = r\cos\theta,\ \ y = r\sin\theta$$

gives

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r \, dr d\theta$$

$$= \int_0^{2\pi} -1/2 e^{-r^2} \Big/_0^{\infty} \, d\theta$$

$$\int_0^{2\pi} 1/2 \, d\theta$$

$$= \pi$$

Thus

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}.$$

Does it work in other cases?

Which other functions can be integrated by this method?

Consider the integral

$$I = \int_{-\infty}^{\infty} f(x)\, dx.$$

Proceeding as before

$$I^2 = \left(\int_{-\infty}^{\infty} f(x)\, dx \right) \left(\int_{-\infty}^{\infty} f(y)\, dy \right)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) f(y)\, dx dy$$

In order to continue the method further there must exist functions g and h such that

$$f(x)f(y) = g(x^2 + y^2)h(y/x).$$

We will then have

$$I^2 = \int_0^{2\pi} \int_0^{\infty} g(r^2)h(\tan\theta)r\,dr d\theta$$

$$= \left(\int_0^{\infty} g(r^2)r\,dr\right)\left(\int_0^{2\pi} h(\tan\theta)d\theta\right)$$

$$= \frac{1}{2}\left(\int_0^{\infty} g(x)dx\right)\left(\int_0^{2\pi} h(\tan\theta)d\theta\right).$$

We now need to compute these integrals. This will not always be possible.

Which functions f admit a decomposition

$$f(x)f(y) = g(x^2 + y^2)h(y/x)$$

where the functions $g(x)$ and $h(\tan\theta)$ have elementary antiderivatives?

Theorem 1. *Suppose* $f : (0, \infty) \mapsto \mathbf{R}$ *satisfies an equation of the form*

$$f(x)f(y) = g(x^2 + y^2)h(y/x), \quad x, y > 0.$$

Also assume there exist d *and* $A > 0$ *such that*

$$\lim_{x \to 0^+} x^{-d} f(x) = A. \qquad (IC)$$

(say $f(x) \sim Ax^d$ *as* $x \to 0^+$*).*

Then f *has the form*

$$f(x) = Ax^d e^{cx^2}.$$

The functions g *and* h *are unique (up to scalar multiples) and are given by*

$$g(x) = A_1 x^d e^{cx},$$

$$h(x) = A_2 \Big(\frac{x}{1 + x^2} \Big)^d$$

where $A_1 A_2 = A$.

Defining
$$I = \int_{-\infty}^{\infty} x^d e^{cx^2} dx$$
we have
$$I^2 = \frac{1}{2^{d+1}} \int_0^{\infty} x^d e^{cx}\, dx \times \int_0^{2\pi} \sin^d t\, dt.$$

Can these integrals be evaluated?

The existence of the first integral requires that $d > -1$ and $c < 0$. The evaluation of the second integral requires that d be an integer. But then the integral
$$\int_{-\infty}^{\infty} x^d e^{cx^2}\, dx$$
can be evaluated in terms of
$$\int_{-\infty}^{\infty} e^{-x^2}\, dx$$
by substitution and integration by parts.

Poisson s trick to integrate e^{-x^2} is of no further use as an integration method!

Proof of the theorem

We assume

$$f(x)f(y) = g(x^2 + y^2)h(y/x). \qquad (1)$$

Firstly, note that $h(1) \neq 0$ otherwise taking $y = x$ in (1) shows $f \equiv 0$, contradicting the initial condition. We suppose wlog $h(1) = 1$.

Taking $y = x$ in (1) we get

$$f^2(x) = g(2x^2).$$

Substituting for g in terms of f in (1)

$$f(x)f(y) = f^2(\sqrt{(x^2 + y^2)/2})h(y/x) \qquad (2)$$

Define

$$F(x) \equiv x^{-d} f(x),$$

so that

$$\lim_{x \downarrow 0} F(x) = A > 0.$$

Writing (2) in terms of F, we have

$$(2xy)^d F(x)F(y) = (x^2 + y^2)^d \times$$

$$F^2(\sqrt{(x^2 + y^2)/2})h(y/x) \quad (3)$$

Set $y = tx$ to get

$$(2t)^d F(x)F(tx) = (1+t^2)^d F^2(x\sqrt{(1 + t^2)/2})h(t).$$

Letting $x \to 0$,

$$(2t)^d A^2 = (1 + t^2)^d A^2 h(t)$$

and we conclude that

$$h(t) = \left(\frac{2t}{1 + t^2}\right)^d.$$

Substituting this into (3) gives

$$F(x)F(y) = F^2\left(\sqrt{\frac{x^2 + y^2}{2}}\right).$$

Note that this equation implies F is non-negative. We now show that F is *non-vanishing*.

Suppose $F(y_0) = 0$ for some y_0. Then

$$F(x)F(y_0) = F^2\left(\sqrt{\frac{x^2 + y_0^2}{2}}\right)$$

implies $F(y) = 0$ for all $y > y_0/\sqrt{2}$. Iterating this step, we see that $F \equiv 0$ contradicting IC. Thus F is *strictly positive*. Define

$$\alpha(x) = \log F(\sqrt{x}) - \log A,\ x > 0$$

$$\alpha(0) = 0.$$

Then α is continuous at 0 and satisfies

$$\alpha(x) + \alpha(y) = 2\alpha\Big(\frac{x + y}{2}\Big),\ \ x, y > 0. \qquad (4)$$

Setting $y = 0$, we have $\alpha(x) = 2\alpha(x/2)$ and substituting this into (4) gives

$$\alpha(x) + \alpha(y) = \alpha(x + y).$$

This implies that α has the form $\alpha(x) = cx$.

Solving for F in

$$\alpha(x) = \log F(\sqrt{x}) - \log A$$

we get

$$F(x) = Ae^{cx^2}.$$

Thus

$$f(x) = x^d A e^{cx^2}$$

and the theorem is proved.

Changing the initial condition

What happens when f decays or explodes at *exponential* or *logarithmic* rate at zero?

Theorem 2. *Suppose* $f(x) \sim e^{c/x^d}$ *as* $x \to 0^+$ *for some* $c \neq 0$ *and* $d > 0$*. Then there exist no functions* g *and* h *such that*

$$f(x)f(y) = g(x^2 + y^2)h(y/x).$$

Proof. As in the proof of Theorem 1, we have

$$f(x)f(y) = f^2\Big(\sqrt{\frac{x^2+y^2}{2}}\Big)\frac{h(y/x)}{h(1)}.$$

Set $y = tx$ to get

$$f(x)f(tx) = f^2\Big(x\sqrt{\frac{1+t^2}{2}}\Big)\frac{h(t)}{h(1)}.$$

Define

$$F(x) = e^{-c/x^d} f(x).$$

Writing the previous equation in terms of F, fixing $t > 0$ and letting x approach 0, gives

$$h(t) = \lim_{x\to 0+} h(1) \exp\Big\{\frac{c}{x^d} H(t)\Big\} \tag{5}$$

where

$$H(t) = \frac{2}{(\frac{1+t^2}{2})^{d/2}} - \frac{1}{t^d} - 1.$$

Note that $H(t)$ is *negative* if

$$t < t_0 = \frac{1}{\sqrt{2^{\frac{2}{d}+1} - 1}}.$$

In the case $c < 0$, equation (5) implies that for t in this range, $h(t)$ *fails to exist*.

In the case when $c > 0$ ($f(x)$ *blows up as* $x \to 0$), equation (5) implies that $h(t) \equiv 0$ for $t < t_0$. Recall

$$f(x)f(y) = f^2\left(\sqrt{\frac{x^2+y^2}{2}}\right)\frac{h(y/x)}{h(1)}.$$

Fix a point x such that $f(x) \neq 0$ and let $y \to 0$. We conclude that $f(y) \equiv 0$ for y in a neighborhood of 0, a contradiction.

Theorem 3. *Suppose*

$$f(x) \sim \log^d x, \;\; x \to 0^+ \tag{6}$$

for some $d \neq 0$. *Then there are no functions* g *and* h *satisfying*

$$f(x)f(y) = g(x^2 + y^2)h(y/x). \tag{7}$$

Proof. The preceding argument shows that if both (6) and (7) hold then

$$h(t) = h(1) \lim_{x \downarrow 0} \frac{(\log^d x)\log^d(tx)}{\log^{2d}(xT)}$$

$$= h(1) \lim_{x \downarrow 0} \frac{(\log^d x)\Big(\log x + \log t\Big)^d}{\Big(\log x + \log T\Big)^{2d}}$$

$$= h(1)$$

where $T = \sqrt{\frac{1+t^2}{2}}$.

Thus we have

$$f(x)f(y) = f^2\Big(\sqrt{\frac{x^2+y^2}{2}}\Big)h(1) \qquad (8)$$

This is inconsistent with the IC

$$f(x) \sim \log^d x, \;\; x \to 0^+ \qquad (9)$$

To see this, substitute $y = x^2$ in (8) to get

$$f(x)f(x^2) = f^2\Big(x\sqrt{\frac{1+x^2}{2}}\Big)h(1). \qquad (10)$$

Set $F(x) = (\log^{-d} x)f(x)$, then F has a non-zero limit at 0. Write (10) in terms of F and let $x \downarrow 0$. Then (9) implies

$$\left[\frac{(\log x)\log(x^2)}{\log^2\left(x\sqrt{\frac{1+x^2}{2}}\right)}\right]^d \to 1.$$

This is incorrect as the above limit is, in fact, equal to 2^d. This contradiction proves the theorem.

Theorem 4. *Suppose functions f, g, h satisfy equation (1) and f is strictly positive and locally integrable in some interval* $(0, \epsilon)$*. Then there exist constants A, A_1, A_2, d and c such that*

$$f(x) = Ax^d e^{cx^2},$$

$$g(x) = A_1 x^d e^{cx},$$

$$h(x) = A_2 \Big(\frac{x}{1+x^2}\Big)^d$$

where $A_1 A_2 = A$.

Proof. We prove the theorem under the assumption that f is C^1 in $(0, \epsilon)$. The full result is an easy extension using Schwartz distributions.

As in the proof of Theorem 1, we may assume that $h(1) = 1$. As before, we derive the equation

$$f(x)f(y) = f^2\Big(\sqrt{\frac{x^2+y^2}{2}}\Big)h(y/x),\ x,y > 0. \tag{11}$$

Setting $y = tx$ gives

$$f(x)f(tx) = f^2\Big(x\sqrt{\frac{1+t^2}{2}}\Big)h(t),\ t,x > 0. \tag{12}$$

Replace x by $\sqrt{x/t}$ and denote

$$s = \frac{t+1/t}{2},\ f(x) = r(x^2)$$

Assuming xt and $x/t \in (0,\epsilon^2)$, we may write (12) as

$$h(t) = \frac{r(xt)r(x/t)}{r^2(xs)}. \tag{13}$$

Taking the log in (13), differentiating wrt x and multiplying by x gives

$$R(xt) + R(x/t) = 2R(xs) \qquad (14)$$

where

$$R(x) \equiv \frac{xr'(x)}{r(x)},\ x \in (0, \epsilon^2). \qquad (15)$$

Set $xt = a$ and $x/t = b$. Then $xs = (a + b)/2$ and (14) gives

$$R(a) + R(b) = 2R\Big(\frac{a+b}{2}\Big).$$

Since R is continuous, we that conclude R is a *linear* function. Writing

$$R(x) = cx + d/2$$

and solving for r in (15) gives (up to a multiplicative constant)

$$r(x) = x^{d/2} e^{cx},\ x \in (0, \epsilon^2).$$

Substituting this expression into (13), we obtain

$$h(t) = \Big(\frac{2t}{1+t^2}\Big)^d,\ t > 0.$$

Equation (11) now reads

$$f(x)f(y) = f^2\Big(\sqrt{\frac{x^2+y^2}{2}}\Big)\Big(\frac{2xy}{x^2+y^2}\Big)^d, \; x, y > 0.$$

Define $F(x) = x^{-d}f(x)$. Then F satisfies the equation

$$F(x)F(y) = F^2\Big(\sqrt{\frac{x^2+y^2}{2}}\Big), \; x, y > 0.$$

Arguing as in the proof of Theorem 1, we conclude that $F(x) = Ae^{cx^2}$ for some constant A.

Thus $f(x) = Ax^d e^{cx^2}$ as required.

In conclusion

We have studied the existence of solutions to the functional equation

$$f(x)f(y) = g(x^2 + y^2)h(y/x), \;\; x, y > 0.$$

We showed

(i) if $f(x)$ asymptotic to x^d at zero, then g and h exist if and only if

$$f(x) = Ax^d e^{cx^2}.$$

This conclusion holds without the asymptotic hypothesis, provided f is locally integrable and strictly positive in a neighborhood of zero.

(ii) If f has *exponential* or *logarithmic* behavior at zero, then g and h *do not exist*.

Finally...

There is nothing sacred about polar coordinates. We could consider other changes of coordinates $(x, y) \mapsto (u, v)$ and try to characterize the class of functions f for which there exist a pair of functions g and h such that

$$f(x)f(y) = g(u)h(v).$$