Generalized divergence operators and transformations of measure

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1. The measure transformation problem

Given a Borel measure ν on a vector space E and a measurable transformation $T : E \mapsto E$, define the induced measure ν_T by

$$\nu_T(B) = \nu(T^{-1}(B)).$$

Find conditions on T *such that the* $\nu_T << \nu$, *and compute the Radon-Nikodym derivative* $d\nu_T/d\nu$ *when these conditions hold.*

Examples

1.**Classical theorem**. Let ν be Lebesgue measure on $\mathbf{R}^d$. If $T : U \subset \mathbf{R}^d \mapsto \mathbf{R}^d$ is an injective map and T^{-1} is C^1, then $\nu_T << \nu$ and

$$\frac{d\nu_T}{dx} = |\det DT^{-1}(x)|$$

2. Cameron-Martin Theorem. Consider an abstract Wiener space (H, E, ν), i.e E is a separable Banach space, H is a densely embedded Hilbert subspace of E and ν is the completion to E of the canonical cylinder set measure on H. Let $h \in H$ and denote by ν_h the measure $\nu(. + h)$. Then $\nu_h \sim \nu$ and

$$\frac{d\nu_h}{d\nu}(x) = \exp -\Big\{ < h, x >_H + 1/2||h||_H^2 \Big\}$$

More generally, there is the following result

3. Theorem (H-H Kuo). In an abstract Wiener space (H, E, ν) let T be a C^1 diffeomorphism of E. Suppose I has the form

$$T = I + K$$

where I is the identity on E and K is a C^1 map from E to E^*. Then $\nu_T \sim \nu$ and

$$\frac{d\nu_T}{d\nu} \circ T(x) = |\det_H DT(x)|^{-1} \times$$

$$\exp\Big\{(K(x), x) + 1/2||K(x)||_H^2\Big\}.$$

Remark: In this context E^* is considred to be a subset of E under the maps

$$E^* \overset{i^*}{\mapsto} H^* \overset{\sim}{=} H \overset{i}{\mapsto} E$$

4. Girsanov Theorem. Let w be a Wiener process (defined on the time interval $[0,T]$) and let ν denote the Wiener measure on the space of paths $E = C_0([0,T];\mathbf{R})$. Let h be a real-valued bounded adapted process and define

$$v_t = w_t + \int_0^t h_s\, ds, \;\; t \in [0,T].$$

Then v is a Wiener process with respect to the measure γ, where γ is the measure on E defined by

$$\frac{d\gamma}{d\nu}(w) = \exp -\Big\{ \int_0^T h_s dw_s + \frac{1}{2}\int_0^T h_s^2\, ds \Big\}.$$

2. Generalized Divergence Operators

Let ν be a Borel measure on a vector space E. Let U denote a distinguished subset of the class of functions from E to E.

Definition. A linear operator $L : U \mapsto L^2(\nu)$ is a *Generalized Divergence Operator* (GDO) for ν if the following relation holds for all test functions ϕ (real-valued C^1 functions on E such that ϕ and $D\phi$ are bounded), and all $h \in U$ such that both sides exist

$$\int_E D\phi(x)h(x)d\nu = \int_E \phi(x)Lh(x)d\nu.$$

Examples

Theorem (Gaveau-Trauber). The Ito integral operator

$$L_1 h(w) = \int_0^T h_s dw_s$$

is a GDO for the Wiener measure.

The domain of L_1 is the set of all non-anticipating absolutely continuous processes h such that

$$\int_0^T h_s'^2 ds < \infty, \ \ a.s$$

Let (H, E, Γ) be an abstract Wiener space. The operator

$$L_2 h(x) = (h(x), x) - trace_H Dh(x)$$

is a GDO for Γ. The domain of this operator is the set of C^1 maps from E to E^*.

Following Nualart and Pardoux, the operator L_2 can be used to define a version of the stochastic integral with *anticipating* integrands (Skorohod integral).

We have the following easily verifiable result:

Lemma. Let $h \in U \cap L^2(\nu), \psi : E \mapsto \Re,^2 (\nu) \cap C^1$. Suppose $\phi h \in U$. Then

$$L(\psi h)(x) = \psi(x) Lh(x) - D\psi(x) h(x), \quad a.s.(\nu)$$

3. A Homotopy construction

Let (L, U) be a GDO for ν, and let $T : E \mapsto E$ be a map of the form $I + K$ where I is the identity on E and $K \in U$. Define

$$T_t \equiv I + tK, \ \ t \in [0, 1]$$

Suppose T_t is invertible for all $t \in [0, 1]$.

Note that $\nu_{T_t} << \nu$ if and only if there exists a corresponding family X_t of Radon-Nikodym derivatives $d\nu_{T_t}/d\nu$.

Let ϕ be a test function on E. Then

$$\int_E \phi \circ T_t(x) d\nu = \int_E \phi d\nu_t = \int_E \phi X_t d\nu.$$

Replacing ϕ by $\phi \circ T_t^{-1}$, we have

$$f(t) \equiv \int_E \phi \circ T_t^{-1}(x) X_t(x) d\nu = \int_E \phi(x) d\nu. \quad (*)$$

Thus $f'(t) \equiv 0$.

Formal differentiation wrt t under the first integral in (*) gives

$$\int_E \Big\{ D\phi(T_t^{-1}(x))(d/dtT_t^{-1}(x))X_t(x)$$

$$+\phi \circ T_t^{-1}(x) d/dt X_t(x) \Big\} d\nu = 0$$

We rewrite the first term in the integrand using the relation

$$D\phi(T_t^{-1}(x))d/dtT_t^{-1}(x) = -D(\phi\circ)(x)K\circ T_t^{-1}(x)$$

to get

$$\int_E \Big\{ \phi \circ T_t^{-1}(x) d/dt X_t(x)$$

$$-D(\phi \circ T_t^{-1})(x) K \circ T_t^{-1}(x) X_t(x) \Big\} d\nu = 0.$$

Now assume that $(K \circ T_t)X_t \in U$, for all $t \in [0,1]$. We use the defining property of L to write this in the form

$$\int_E \phi \circ T_t^{-1}(x)\Big\{d/dtX_t(x) - L[(K \circ T_t^{-1})X_t](x)\Big\}d\nu = 0.$$

This will hold for all test functions ϕ *if and only if* X_t satisfies the differential equation

$$d/dtX_t(x) - L[(K \circ T_t^{-1})X_t](x) = 0$$

Applying the Lemma to the second term yields

$$d/dtX_t(x) = X_t(x)L[K \circ T_t^{-1}](x)$$

$$-DX_t(x)K \circ T_t^{-1}(x)$$

Denote $X_t(x)$ by $X(t,x)$, $\partial X/\partial t$ by X_1, $\partial X/\partial x$ by X_2, and substitute $x = T_t(y)$ to obtain

$$X_1(t, T_t(y)) = X(t, T_t(y))L[K \circ T_t^{-1}](T_t(y))$$

$$-X_2(t, T_t(y))K(y)$$

Since $K = dT_t/dt$, this is equivalent to

$$d/dt X(t, T_t(y)) = X(t, T_t(y)) L[K \circ T_t^{-1}](T_t(y))$$

Solving this equation with the initial condition $X(0, .) = 1$ gives

$$X(t, T_t(y)) = \exp\left\{ \int_0^t L[K \circ T_s^{-1}](T_s(y)) \right\} ds$$

Hence

$$X(t, x) = \exp\left\{ \int_0^t L[K \circ T_s^{-1}](T_s \circ T_t^{-1}(x)) \right\} ds$$

Finally

$$\frac{d\nu_T}{d\nu}(x) = \exp\left\{ \int_0^1 L[K \circ T_s^{-1}](T_s \circ T_t^{-1}(x)) \right\}.$$

4. Transformation formulae

We give some examples of the formula

$$\frac{d\nu_T}{d\nu}(x) = \exp\Big\{ \int_0^1 L[K \circ T_s^{-1}](T_s \circ T_t^{-1}(x)) \Big\}.$$

1. If ν is a Gaussian measure in abstract Wiener space, then taking L to be L_2 we obtain the RND in Kuo s theorem

$$\frac{d\Gamma_T}{d\Gamma} \circ T(x) = |\det_H DT(x)|^{-1} \times$$

$$\exp\Big\{ (K(x), x) + 1/2 ||K(x)||_H^2 \Big\}.$$

2. Let ν be Wiener measure on the space of paths $C_0([0, T]$. Taking L to be the Ito integral operator L_1, we get the Girsanov density

$$G(w) = \exp -\Big\{ \int_0^T h_s dw_s + 1/2 \int_0^T h_s^2\, ds \Big\}.$$

In principle, all the steps in the above argument are *reversible*. Suppose we have a measure ν on E, a GDO operator (L, U) for ν, and a map $T = I + K$ of E such that $K \in U$ and $I + tK$ is invertible for all $t \in [0, 1]$.

Then we can seek to obtain a transformation theorem by *defining*

$$X(t, x) = \exp\left\{ \int_0^t L[K \circ T_s^{-1}](T_s \circ T_t^{-1}(x))\right\} ds$$

then reverse the argument to obtain

$$\int_E \phi \circ T_t^{-1}(x) X_t(x) d\nu = \int_E \phi(x) d\nu$$

This will prove that $\nu_{T_t} << \nu$ for all $t \in [0, 1]$ and

$$\frac{d\nu_{T_t}}{d\nu} = X(t, .)$$

This method was used by Smolyanov and Weizsacker to prove the following non-Gaussian version of the Cameron-Martin theorem.

Theorem. Let ν be a Borel measure on a vector space E. Suppose that there exists an L^1 random variable X_h such that for all test functions ϕ

$$\int_E D_h\phi \, d\nu = \int_E \phi X_h \, d\nu$$

Assume there exists $\delta > 0$ such that the following integral exists

$$\int_{-\delta}^{\delta} |X_h(x+uh)| du.$$

Then ν is quasi-invariant under translation by h and

$$\frac{d\nu_h}{d\nu}(x) = \exp\Big\{ - \int_0^1 X_h(x+uh) du \Big\}.$$