

The Gauss-Bonnet Theorem for Vector Bundles

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1. The theorem

We aim to prove the result

Theorem. *Let E denote a real orientable Riemannian vector bundle of even rank $n = 2k$ over a closed compact manifold M of dimension n . Then*

$$\left(\frac{-1}{2\pi}\right)^k \int_M Pf(\Omega) = \chi(E)$$

where Pf denotes the Pfaffian function acting on square skew-symmetric matrices, Ω the curvature 2-form with respect to any metric connection on E , and $\chi(E)$ the Euler characteristic of E .

2. The topological Euler class

Let E denote an oriented Riemannian vector bundle of even rank p over M . Let $i : M \mapsto E$ denote embedding as the 0-section.

Thom Isomorphism Theorem. *There exists a unique element $u \in H_c^p(E)$, known as the Thom class of E , such that*

$$\int_{E_x} u = 1$$

for each fiber E_x of E .

Define the (topological) *Euler class* e_t of $E \in H^p(M)$ by $i^*(u)$ and in the case $p = n$, the *Euler characteristic* χ of E by

$$\chi = \int_M e_t.$$

Let $X : M \mapsto E$ denote a generic vector field.
Then

$$\begin{aligned}
 X(M) \cap i(M) &= \{x_1, \dots, x_r\}. \\
 \chi &= \int_{i(M)} u = \lim_{t \rightarrow 0^+} \int_{tX(M)} u \\
 &= \lim_{t \rightarrow \infty} \int_{tX(M)} u = \sum_{i=1}^r (+/-) \int_{E_{x_i}} u \\
 &= \sum_{i=1}^r I_{x_i}(X).
 \end{aligned}$$

Theorem. *If there exists a non-vanishing vector field $X : M \mapsto E$, then $\chi = 0$.*

Uniqueness of the Thom class

Suppose there exist $u_1, u_2 \in H_c^p(E)$ satisfying the conditions of the Thom class.

Define $u = u_1 - u_2$. Let U be a generic p -cycle in E . Then $U \cap i(M) = \{x_1, \dots, x_r\}$.

$$\int_U u = \lim_{t \rightarrow \infty} \int_{tU} u = \sum_{i=1}^r (+/-) \int_{E x_i} u = 0.$$

This then holds for all p -cycles U .

Using a compactification argument and De Rham's theorem, we can now show that $u = d\omega$ where ω has compact support, i.e. $u = 0 \in H_c^p(E)$.

Suppose E_1 and E_2 are isomorphic vector bundles over M .

$$\begin{array}{ccc}
 & & T \\
 E_1 & \xrightarrow{\quad} & E_2 \\
 \uparrow i & \nearrow j & \\
 M & &
 \end{array}$$

It is clear that $T^*(u(E_2)) \in H_c^p(E)$, Furthermore

$$\int_{E_1, x} T^*(u(E_2)) = \int_{T(E_1, x)} u(E_2) = \int_{E_2, T(x)} u(E_2) = 1.$$

By uniqueness, we conclude $T^*(u(E_2)) = u(E_1)$. Hence

$$\begin{aligned}
 e_t(E_2) &= j^*(u(E_2)) = i^*(T^*(u(E_2))) \\
 &= i^*(u(E_1)) = e_t(E_1)
 \end{aligned}$$

That is, e_t is well-defined on the *isomorphism class* of vector bundles over M .

3. The geometrical Euler class

Let ∇ denote any metric connection on E . We define the pfaffian of the corresponding curvature form $Pf(\Omega) \in H^p(M)$.

Theorem. *$Pf(\Omega)$ is closed. Furthermore, $[Pf(\Omega)] \in H^p(M)$ is independent of the particular choice of connection on E used in its construction.*

The geometrical Euler class e_g of M is defined by

$$e_g = \left[\left(\frac{-1}{2\pi} \right)^{p/2} Pf(\Omega) \right].$$

We have now introduced two cohomology classes, e_t and e_g . We will prove the theorem: $e_t = e_g$.

Since, in the case when $p = n$, we have

$$\chi = \int_M e_t = \int_M e_g$$

this will complete the proof.

4. The proof

We first note that both classes (which we denote by e below) share the following two properties

Whitney duality: Let $E = E_1 \oplus \cdots \oplus E_r$ be a direct sum of oriented even-dimensional bundles. Then

$$e(E) = e(E_1) \wedge \cdots \wedge e(E_r)$$

Naturality: Let $f : N \mapsto M$ be a C^∞ map and consider the pull-back bundle $f^*(E)$ over N

$$\begin{array}{ccc} f^*(E) & & E \\ \downarrow & & \downarrow \\ N & \xrightarrow{\quad f \quad} & M \end{array}$$

Then $e(f^*(E)) = f^*(e(E))$.

Theorem.

$$e_g = e_t \in H^p(M). \quad (1)$$

Proof. First, assume E is a plane bundle with metric connection ∇ . We give an explicit construction of the Thom class u in terms of ∇ .

Let $\{e_1, e_2\}$ be a (locally defined) oriented orthonormal frame for E and ω_e the upper off-diagonal entry of the corresponding connection 1-form with respect to this frame. Let v_1, v_2 denote the components of E -vectors with respect to the frame $\{e_1, e_2\}$ and r the radial distance in any E_x . Finally, ρ and γ will denote smooth functions from $[0, \infty)$ to \mathfrak{R} with compact support and c a dimensionally-dependent constant, to be defined in the course of the proof.

Consider the following locally defined 2-form on E :

$$u \equiv c\{\rho(r^2)dv_1dv_2 - \rho(r^2)rdr\pi^*(\omega_e) + \pi^*(d\omega)\gamma(r)\}, \quad (2)$$

where $\pi : E \mapsto M$ is the projection map. Note that r and $d\omega$ are intrinsic objects. Changing to another orthonormal frame e with the same orientation as e , related to e by a counterclockwise rotation of θ , yields

$$dv_1dv_2 = dv_1dv_2 + rdr\pi^*(d\theta).$$

However,

$$\omega_e = \omega_e + d\theta.$$

Substituting into (2)

$$u = c\{\rho(r^2)dv_1dv_2 - \rho(r^2)rdr\pi^*(\omega_e) + \pi^*(d\omega)\gamma(r)\},$$

i.e. u is *intrinsic*. We shall define ρ, γ , and c so that u is the Thom class.

First choose γ so that $\gamma(0) = 1$. Since $rdr = v_1dv_1 + v_2dv_2$ and $\pi \circ i = id$, applying i^* to (2) yields

$$i^*(u) = cd\omega \quad (3)$$

Taking the exterior derivative in (2) gives

$$du = c\pi^*(d\omega)dr\{(\gamma'(r) - \rho(r^2)r)\}.$$

Thus a sufficient condition for u to be closed is $\gamma'(r) = \rho(r^2)r$. Together with the condition $\gamma(0) = 1$, this implies

$$\gamma(r) = 1 + \int_0^r \rho(s^2)sd s.$$

We now choose ρ to be any smooth function with support contained in $(0, 1)$ such that

$$\int_0^1 \rho(s^2)sd s = -1$$

and extend ρ and γ by defining them to be 0 on $[1, \infty)$.

It follows from (2) that

$$\begin{aligned}\int_{E_x} u &= c \int_{\mathbf{R}^2} \rho(r^2) dv_1 dv_2 \\ &= c \int_0^{2\pi} d\phi \int_0^\infty \rho(r^2) r dr \\ &= c \int_0^{2\pi} d\phi \int_0^1 \rho(r^2) r dr = -2\pi c.\end{aligned}$$

Thus choosing $c = -1/2\pi$ ensures that u is (a representative of) the Thom class. From (3) we have

$$e_t = i^*(u) = e_g$$

and the theorem is proved for the plane bundle case.

Suppose now that $E = E_1 \oplus \cdots \oplus E_q$ is a sum of orientable plane bundles. Let E_1, \dots, E_q have curvature 2-forms with upper off-diagonal entries $d\omega_1, \dots, d\omega_q$ with respect to (any) metric

connections. Let ∇ be the corresponding direct sum connection on E . Then

$$pf(\Omega) = d\omega_1 \dots d\omega_q \quad (4).$$

Let E_1, \dots, E_q have topological Euler classes e_t^1, \dots, e_t^q . Using (4), the theorem for plane bundles and Whitney duality, we obtain

$$\begin{aligned} e_g(E) &\equiv (-2\pi)^{-q} Pf(\Omega) \\ &= (-d\omega_1/2\pi) \dots (-d\omega_q/2\pi) \\ &= e_g^1 e_g^2 \dots e_g^q \\ &= e_t^1 e_t^2 \dots e_t^q \\ &= e_t(E). \end{aligned}$$

Thus the theorem holds when E is a sum of oriented plane bundles.

We deduce the general case from the following result.

Theorem (Splitting Principle). *Let E denote a real, orientable, even dimensional vector bundle over a manifold M . Then there exists a manifold N and a map $g : N \mapsto M$ such that*

(i) $g^ : H^*(M) \mapsto H^*(N)$ is a monomorphism (i.e. is injective).*

(ii) $g^(E)$ is a sum of orientable plane bundles.*

Suppose now E is an arbitrary orientable vector bundle over M . Applying the above result with E and using the already established coincidence of e_g and e_t on sums of orientable plane bundles and the naturality of e_g and e_t , we have

$$g^*(e_g(E)) = e_g(g^*(E)) = e_t(g^*(E)) = g^*(e_t(E)).$$

The result now follows from the injectivity property of the map g^* .