Divergence Theorems in Path Space

Denis Bell

University of North Florida

1. Motivation

Let M denote a closed d-dimensional Riemannian manifold. A basic result in differential geometry is the *Divergence Theorem*:

Theorem. Let Z be a C^1 vector field on M. Then for all test functions Φ on M, we have

$$\int_M Z(\Phi)dx = \int_M \Phi Div(Z)dx$$

where $Div(Z)$ (the *divergence* of Z)is given in local co-ordinates by

$$Div(Z) = \frac{-1}{\sqrt{\det g}} \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(a_i \sqrt{\det g})$$

where g is the metric tensor and $\sum_{i=1}^{d} a_i \frac{\partial}{\partial x_i}$ is a local representation of Z.

Divergence $\Rightarrow$ Laplacian $\equiv -Div\nabla$ $\Rightarrow$heat kernel, spectral theory, harmonic functions, harmonic forms, Hodge theory, index theory,. . .

Generalizing the Divergence Theorem to infinite-dimensional manifolds is a highly non-trivial problem. In particular there is no analogue of the volume form dx in this setting. We seek to replace dx by a finite Borel measure γ.

For example, consider the manifold X of continuous paths $\{w : [0,T] \mapsto \mathbf{R} / w(0) = 0\}$ and let γ be the *Wiener measure* on X.

Suppose h is a path with finite energy

$$\int_0^T |h'(t)|^2 dt.$$

By the Girsanov theorem

$$E[\Phi(w+sh)] = E\Big[\Phi(w) \exp\Big(s \int_0^T h' dw - \frac{s^2}{2} \int_0^T h'^2 dt\Big)\Big].$$

Differentiating wrt s and setting $s = 0$ gives

$$E[h(\Phi)(w)] = E\Big[\Phi(w) \int_0^T h' dw\Big].$$

This relation

$$E[h(\Phi)(w)] = E\Big[\Phi(w)\int_0^T h'dw\Big]$$

shows that the *divergence* of h (wrt γ) exists and is given by

$$Div(h) = \int_0^T h'dw.$$

In general, the divergence of a C^1 map $h : X \mapsto X$ *will not exist*, e.g. $h(w) = w$. Note that

$$\int_0^T w'dw$$

makes no sense.

Problem Given an infinite-dimensional manifold X equipped with a measure γ, construct a class of vector fields on X for which divergence exists and compute the divergence of these vector fields.

2. Admissible vector fields

Let X be a Banach manifold and γ a finite Borel measure on X.

Definition. A vector field Z on X is *admissible* if there exists an L^1 function $Div(Z)$ such that for all test functions Φ on X

$$\int_X Z(\Phi) d\gamma = \int_X \Phi Div(Z) d\gamma.$$

Example. The classical Wiener space.

X is the space of continuous paths

$$\Big\{w : [0,T] \mapsto \mathbf{R}^n / w(0) = 0\Big\}.$$

with the n-dimensional Wiener measure.

Define H to be the *Cameron-Martin space*

$$\Big\{h \in X \Big/ \int_0^T |h'_t|^2 dt < \infty\Big\}.$$

Theorem. Let $h : X \mapsto H$ be a bounded random *adapted* path, i.e.

$$h(s) = f(w_u / u \leq s).$$

Then h is admissible and

$$Div(h) = \int_0^T h' \cdot dw.$$

There exists another class of admissible vector fields on the Wiener space.

Theorem. Let a be a continuous adapted process taking values in $so(n)$ (the set of $n \times n$ skew-symmetric matrices). Define

$$Z = \int_0^{\cdot} a dw.$$

Then Z is admissible and $Div(Z) = 0$.

Proof. Define

$$w_t^{\epsilon} = \int_0^t e^{\epsilon a} dw.$$

Then $w^0 = w$ and $\frac{dw^{\epsilon}}{d\epsilon}/_{\epsilon=0} = Z$. By the infinitesimal rotation-invariance of Wiener measure w^{ϵ} has the same law as w. Hence

$$E[\Phi(w^{\epsilon})] = E[\Phi(w)]$$

Differentiating with respect to ϵ and setting $\epsilon = 0$ gives

$$E[D\Phi(w)Z] = 0.$$

The use of the above result in the present context is a fundamental insight of Bruce Driver.

Combining the previous two theorems, we see that processes of the form

$$\int_0^{\cdot} a dw + \int_0^{\cdot} b dt$$

where a is a continuous adapted $so(n)$-valued process and b is a continuous adapted $\mathbf{R}^n$-valued process, are admissible.

The space of such processes will be called the *Cameron-Martin-Driver space*.

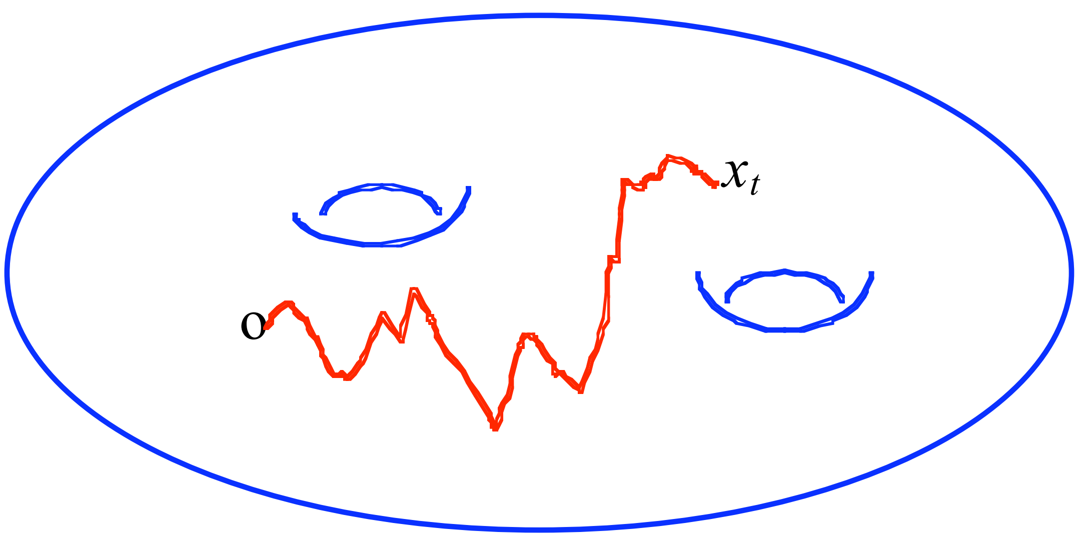
3. Measures induced by stochastic differential equations

Let M denote a closed d-dimensional manifold and $A_1, \ldots, A_n$ smooth vector fields on M. Let o be a fixed point in M.

Consider the (Stratonovich) SDE

$$dx_t = \sum_{i=1}^{n} A_i(x_t) \circ dw_i, \ \ t \in [0, T]$$
$$x_o = o$$

where $(w_1, \ldots, w_n)$ is a Euclidean Wiener process.



We study the manifold of continuous paths

$$X = \Big\{\sigma : [0,T] \mapsto M / \sigma_0 = o\Big\}$$

equipped with the measure γ where γ is the law of the diffusion x.

The tangent space $T_x X$ is defined to be the set of paths $V : [0,T] \mapsto TM$ such that

$$V_t \in T_{x_t} M, \; \forall t \in [0,T].$$

The objective is to construct a class of admissible vector fields on X.

There are two approaches to this problem:

1. The *lifting* approach to the *endpoint* problem (Malliavin, 1976). Recall the SDE

$$dx = \sum_{i=1}^{n} A_i(x_t)dw_i,\ t \in [0, T].$$

Let γ_T denote the law of of x_T. Malliavin established the admissibility of C^1 vector fields on the space (M, γ_T).

The underlying idea is to lift the problem to the Wiener space by the map $w \mapsto x_T$. This approach works under very general nondegeneracy conditions on $A_1, \ldots, A_n$ (Hörmander condition and weaker).

2. The *rolling* approach (Driver, 1991) based on the *stochastic development* map. This method produces admissible vector fields on the whole path space (X, γ) but requires the *ellipticity* condition: $A_1, \ldots, A_n$ span TM at every point in M.

Goals:

1. Construct admissible vector fields on X by the *lifting* method, in the elliptic setting.

2. Extend this method so that it applies to *degenerate* diffusions.

Both these goals have been realized. The first will be described today.

The statement of the main result requires the introduction of some geometric structure associated with the diffusion process

$$dx = \sum_{i=1}^{n} A_i(x_t) \circ dw_i.$$

Assume the vector fields $A_1, \ldots, A_n$ span TM at every point of M.

Then $\{A_i\}$ induce a Remannian metric g on M, defined as follows: let

$$A_i = a_{ir} \partial / \partial x_r$$

be a local representation of $A_i, 1 \leq i \leq n$ (note here and from this point on we use the summation convention).

The metric tensor $[g_{jk}]$, is defined by

$$g^{jk} = a_{ij} a_{ik},\ 1 \leq j, k \leq d.$$

Let ∇ denote the corresponding Levi-Civita connection (the unique metric compatible torsion-free connection on TM).

Define a set of 1-forms $\omega^{jk}, 1 \leq j,k \leq n$ on M by

$$\omega^{jk}(.) =< \nabla_{A_j} A_k, . > - < \nabla_. A_j, A_k >$$

and functions $b^{jk}, 1 \leq j,k \leq n$ on M by

$$b^{jk} = \frac{1}{2}\Big(< L_{ji} A_i, A_k > - < L_{ij} \nabla_{A_k} A_i > -$$

$$< \nabla_{A_j} A_k, \nabla_{A_i} A_i > + < \nabla_{A_p} A_i, A_k >< \nabla_{A_j} A_p, A_i > \Big)$$

where

$$L_{ij} \equiv \nabla_{A_i} \nabla_{A_j} - \nabla_{\nabla_{A_i} A_j}.$$

Denote by H the *Cameron-Martin space* of continuous paths

$$\Big\{ r : [0,T] \mapsto \mathbf{R}^n : r_0 = 0, \int_0^T |r_t'|^2 dt < \infty \Big\}.$$

Theorem. Let $r = (r_1, \ldots, r_n)$ be a path in H. Define $h_i, 1 \leq i \leq n$ by the following system of SDE s

$$dh_i = \Big[\omega^{ji}(\circ dx_t) + b^{ji}(x_t)dt\Big]h_j(t) + r_i' dt$$

$$h_i(0) = 0.$$

Then the vector field Z on $C_o(M)$ defined by

$$Z_t \equiv A_i(x_t)h_i(t), \; t \in [0, T]$$

is admissible and

$$Div(Z) = \int_0^T \Big(r_i' + \frac{1}{2} < Ric(Z_t), A_i(x_t) > \Big)dw_i.$$

Note: The above conclusion is similar to that of Driver but the construction of Z is different. In Drivers work admissible vector fields are obtained by parallel translation along x of Cameron-Martin paths in T_oM.

4. Outline of the proof

Let g denote the *Ito map* $w \mapsto x$ defined by the SDE $dx = A_i(x_t) \circ dw_i$.

The idea is to construct a vector field Z on X that lifts to an *admissible* vector field r on the Wiener space $C_0(\mathbf{R}^n)$.

The lifting property means that the following diagram commutes

$$
\begin{array}{ccc}
 & dg & \\
TC_0(\mathbf{R}^n) & \rightarrow & TX \\
 & & \\
r \uparrow & & \uparrow Z \\
 & & \\
C_0(R^n) & \rightarrow & X \\
 & g &
\end{array}
$$

Following Driver, we require

$$r = \int_o^{\cdot} a dw + \int_0^{\cdot} b dt$$

where a is an adapted process with values in $so(n)$ and b is an adapted process in $\mathbf{R}^n$.

Let Φ be a test function in $C_0(M)$. Then

$$E[(Z(\Phi)(x)\Big] = E\Big[r(\Phi \circ g)(w)]$$

$$E[\Phi \circ g(w) Div(r)]$$

$$= E\Big[\Phi(x) E[Div(r)/x]\Big]$$

where Div denotes the divergence operator in the classical Wiener space.

Thus Z is admissible with divergence given by

$$Div(Z)(x) = E[Div(r)/x]$$

$$= E\Big[\int_0^T b dw \Big/ x\Big]$$

where

$$r = \int_o^{\cdot} a dw + \int_0^{\cdot} b dt.$$

Digression: The endpoint problem

Let $g_T(w) = x_T$ and suppose Z is a C^1 vector field on M.

Then r is lift of Z if

$$dg_T(w)r = Z.$$

Now it can be shown that if $X_1, \ldots, X_n$ satisfy Hormander s condition, then

$$dg_T(w) : H \mapsto T_{x_T}M$$

is a.s. surjective (i.e. g_T is a *submersion*). We choose

$$r = dg_T^*(dg_T dg_T^*)^{-1}Z.$$

The operator $dg_T dg_T^*$ is known as the *Malliavin covariance matrix*.

This construction does not work on the path space level.

Theorem. (Lifting Theorem) The process $r : \Omega \times [0,T] \mapsto \mathbf{R}^n$ is a lift of the vector field Z defined by

$$Z_t \equiv h_i(t) A_i(x_t) \tag{1}$$

if and only if r and h are related by the SDE

$$h_k = r_k + \int_0^\cdot < [A_j, A_i], A_k > (x_t) h_j \circ dw_i. \tag{2}$$

If we choose a path h and define r by (2), then r will not generally lie in the CMD space.

Alternatively, we could choose, say, a deterministic $r \in H$, *define* h as the solution to (2) and Z by (1). However, in this case Z will depend explicitly on w and, since w is generically not a function of x, the process h will not be well-defined as a function of x!

$$(w_1, w_2) \mapsto x$$

$$h = h(w) \neq h(x)$$

The answer is construct (r, Z) as a *pair*.

Observe that the is problem is that the diffusion coefficient in

$$h_k = r_k + \int_0^{\cdot} < [A_j, A_i], A_k > (x_t) h_j \circ dw_i \quad (2)$$

is non-tensorial in A_i. We resolve the issue by decomposing the diffusion coefficient into a term that *is* a tensor in A_i and a term that is *skew-symmetric* in the i and k indices, then absorbing the skew-symmetric part into the process r.

$$< [A_j, A_i], A_k > = < \nabla_{A_j} A_i, A_k > - < \nabla_{A_i} A_j, A_k >$$
$$= < \nabla_{A_j} A_i, A_k > - < \nabla_{A_j} X_k, A_i >$$
$$+ < \nabla_{A_j} A_k, A_i > - < \nabla_{A_i} A_j, A_k > .$$

Write

$$G_j^{ik}(t) = \Big(< \nabla_{A_j} A_i, A_k > - < \nabla_{A_j} A_k, A_i > \Big)(x_t)$$

and

$$T^{jk} = < \nabla_{A_j} A_k, \cdot > - < \nabla . A_j, A_k > .$$

Equation (2)

$$h_k = r_k + \int_0^{\cdot} < [A_j, A_i], A_k > (x_t) h_j \circ dw_i$$

can now be written

$$h_k = r_k + \int_0^{\cdot} G_j^{ik} h_j \circ dw_i + \int_0^{\cdot} T^{jk}(A_i) h_j \circ dw_i$$

$$= r_k + \int_0^{\cdot} G_j^{ik} h_j \circ dw_i + \int_0^{\cdot} T^{jk}(\circ dx) h_j \qquad (3)$$

Let $h = h(x)$ denote the solution to

$$h_k = r_k + \int_0^{\cdot} T^{jk}(\circ dx) h_j$$

and define a process ρ by

$$\rho_k = r_k - \int_0^{\cdot} G_j^{ik} h_j \circ dw_i.$$

Then we have

$$h_k = \rho_k + \int_0^{\cdot} G_j^{ik} h_j \circ dw_i + \int_0^{\cdot} T^{jk}(\circ dx) h_j$$

and (3) holds with r replaced by ρ.

Thus $Z \equiv h_i A_i(x)$is a vector field on $C_o(M)$ and ρ is an admissible lift of Z to $C_0(\mathbf{R}^n)$, as required.

Computation of the divergence

Recall that

$$Div(Z) = E[Div(\rho)/x].$$

In order to compute $Div(\rho)$, it is necessary to convert the Stratonovich integral in

$$\rho_k = r_k - \int_0^{\cdot} G_j^{ik} h_j \circ dw_i.$$

into Ito form. This is done via the formula

$$\int_0^t G_j^{ik} h_j \circ dw_i = \int_0^t G_j^{ik} h_j dw_i + \frac{1}{2}\Big[G_j^{ik} h_j, w_i\Big](t).$$

The divergence of the quadratic variation term above yields the Rcci curvature term in the statement of the theorem.

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