Brownian motion and the heat equation

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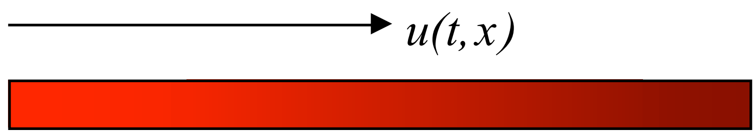
1. The heat equation

Let the function $u(t,x)$ denote the temperature in a rod at position x and time t



Then $u(t,x)$ satisfies the *heat equation*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}, \quad t > 0. \tag{1}$$

It is easy to check that the Gaussian function

$$u(t,x) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}}$$

satisfies (1). Let ϕ be any bounded continuous function and define

$$u(t,x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \phi(y) e^{\frac{-(x-y)^2}{2t}} dy.$$

Then u satisfies (1). Furthermore making the substitution $z = (x-y)/\sqrt{t}$ in the integral gives

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x - y\sqrt{t}) e^{\frac{-z^2}{2}} dz$$

$$\to \phi(x) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-z^2}{2}} dz = \phi(x)$$

as $t \downarrow 0$. Thus

$$u(t,x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \phi(y) e^{\frac{-(x-y)^2}{2t}} dy.$$

$$= E[\phi(X_t)]$$

where X_t is a $N(x,t)$ random variable solves the heat equation

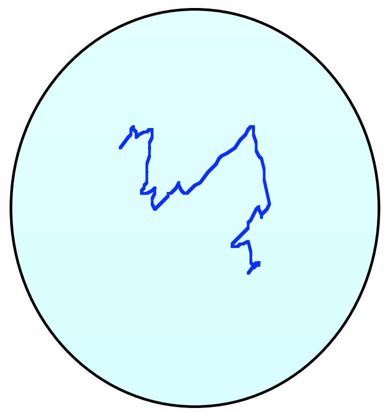
$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}$$

with initial condition $u(0,\cdot) = \phi$.

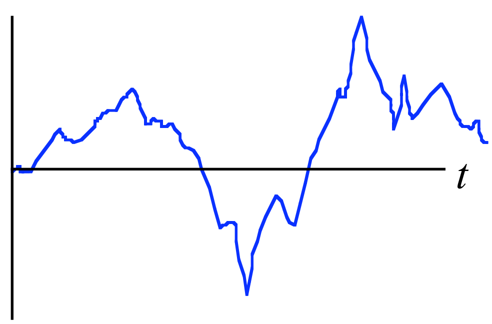
Note: The function $u(t,x)$ is smooth in x for $t > 0$ even if is only *continuous*.

2. Brownian motion

In the nineteenth century, the botanist Robert Brown observed that a pollen particle suspended in liquid undergoes a strange erratic motion (caused by bombardment by molecules of the liquid)



Letting $w(t)$ denote the position of the particle in a fixed direction, the paths w typically look like this



N. Wiener constructed a rigorous mathematical model of Brownian motion in the 1930s.

The mathematical model of Brownian motion (*Wiener process*) satisfies the following axioms:

(i) $w_0 = 0$ and the paths $t \mapsto w_t$ are continuous *a.s.*

(ii) The increments $w_t - w_s$ are independent of $\{w_u / \, u \leq s\}$ for all $t > s$.

(iii) $w_t - w_s$ has a normal distribution with mean 0 and variance $t - s$.

In particular, if we define $w_t^x = w_t + x$ (Wiener process started at x) then by (iii) w_t has a $N(0, t)$ distribution. Hence w_t^x has a $N(x, t)$ distribution, so $u(t, x) = \phi(w_t^x)$ solves the heat equation.

More generally, let **w**$=$ (w$_1, \ldots, w_n$) consist of n independent copies of 1-dimensional Wiener process. Then for any bounded continuous function $\Phi : \mathbf{R}^n \mapsto \mathbf{R}, u(t,x) \equiv E[\phi(\mathbf{w}_t^x)]$ solves the n-dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u$$

with initial condition Φ, where Δ is the Laplacian

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

The Wiener process has many intriguing properies. In particular

1) With probability 1, the path $t \mapsto w_t$ is *non-differentiable* almost everywhere (wrt Lebesgue measure).

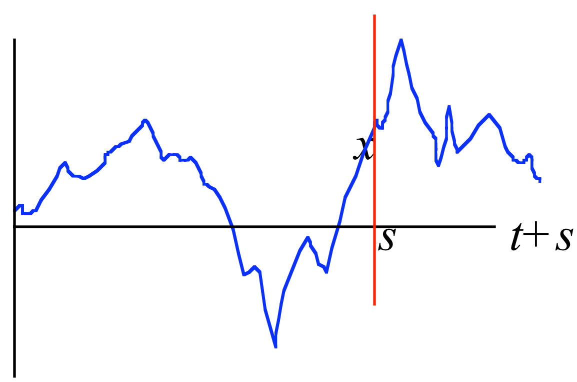
2) $\{w_t\}$ is a *Markov process*.

3. Markov Processes

Definition. Say the stochastic process $\{\xi_t,\ t \geq 0\}$ is a (time-homogeneous) *Markov process* if the conditional distribution of ξ_{t+s} given

$$\{\xi_u, u \leq s, \xi_s = x\}$$

is the same as the distribution of ξ_t^x.



Define the associated *semigroup* $\{P_t,\ t \geq 0\}$

$$(P_t\phi)(x) \equiv E[\phi(\xi_t^x)].$$

acting on bounded continuous functions ϕ.

Theorem. $P_{t+s} = P_s \circ P_t$.

Proof.

$$P_{t+s}\phi(x) = E[\phi(\xi^x_{s+t})]$$

$$= E\Big[E[\phi(\xi^x_{s+t})/\, \xi^x_u, u \le s]\Big]$$

$$= E[(P_t\phi)(\xi^x_s)] = Ps(P_t\phi)(x).$$

Define the *(infinitesimal) generator* of ξ

$$A\phi = \lim_{t \downarrow 0} \frac{P_t\phi - \phi}{t}.$$

Theorem. For ϕ in the domain of A, the function $u(t,x) \equiv (P_t\phi)(x)$ $(= E[\phi(\xi_t^x)])$ solves the Cauchy problem

$$\frac{\partial u}{\partial t} = Au$$

$$u(0,\cdot) = \phi.$$

Proof. Note

$$u(0,x) = E[\phi(\xi_0^x)] = E[\phi(x)] = \phi(x).$$

Furthermore, since $u = P_t\phi$, we have

$$\frac{\partial u}{\partial t} = \lim_{h\downarrow 0} \frac{P_{t+h}\phi - P_t\phi}{h}$$

$$= \lim_{h\downarrow 0} \frac{P_h(P_t\phi) - P_t\phi}{h}$$

$$= A(P_t\phi)$$

$$= Au.$$

4. The Ito integral

The paths of the Wiener process w are of unbounded variation on every time interval $[0, T]$. Nevertheless the *Ito (stochastic) integral*

$$\int_0^T f(s)dw_s$$

can be shown to exist as the limit *in probability* of the sequence of Riemann type sums

$$\int_0^T f(s)dw_s = \lim \sum_{i=1}^{n-1} f(t_i)[w(t_{i+1}) - w(t_i)]$$

with the limit taken over a sequence of partitions $\{0 = t_0 < t_1 < \cdots < t_n = T\}$ of $[0, T]$ with mesh tending to zero.

In order to show this limit exists, we must assume that f satisfies some conditions. In particular:

$f = f(w)$ is *non-anticipating* i.e. $f(t)$ depends only on $\{w_u,\ u \leq t\}$.

Properties of the Ito integral:

1) If $\int_0^T [Ef^2(s)]ds < \infty$ then

$$E\Big[\int_0^T f dw\Big] = 0.$$

The Ito integral does not satisfy the chain rule of classical calculus. It satisfies, instead, a remarkable *second-order* chain rule, known as:

2) *Ito s formula*: Let $\xi_t = \xi_0 + \int_0^t f dw$ and write this as $d\xi = f dw$. If ϕ is a C^2 function then $\eta_t \equiv \phi(\xi_t)$ satisfies

$$d\eta_t = \phi'(\eta_t)d\xi_t + \frac{1}{2}\phi''(\eta_t)f^2(t)dt.$$

We remarked earlier that w is a Markov process. *What is its generator?* - We can compute it using the above properties.

Note that

$$w_t^x = x + \int_0^t 1 dw.$$

Applying Ito s formula with $f = 1$, we have

$$d\phi(w_t^x) = \phi'(w_t^x)dw + \frac{1}{2}\phi''(w_t^x)dt.$$

Writing this in integral form

$$\phi(w_t^x) = \phi(x) + \int_0^t \phi'(w_s^x)dw + \frac{1}{2}\int_0^t \phi''(w_s^x)ds.$$

Taking expectation of each side, using 1) above, differentiating wrt t and setting $t = 0$, we have

$$A\phi(x) = \frac{d}{dt}\Big/_{t=0} E[\phi(w_t^x)]$$

$$= \frac{1}{2}\frac{d}{dt}\Big/_{t=0} \int_0^t E[\phi''(w_s^x)]ds$$

$$= \frac{1}{2}\phi''(x).$$

In view of the first part of the talk, this gives another proof that $u(t,x) = E[\phi(w_t^x)]$ satisfies

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}.$$

This is a very complicated proof of the result, which we derived earlier simply from the fact that w_t^x has a $N(x,t)$ distribution.

However, the stochastic method is valid in a much more general setting.

5. A generalized heat equation

Let a and b be Lipschitz functions. Then it can be shown that the *stochastic differential equation*

$$\xi_t^x = x + \int_0^t a(\xi_s^x)dw + \int_0^t b(\xi_s^x)ds$$

has a unique continuous non-anticipating solution ξ for given initial point x.

Furthermore, ξ is a Markov process. To see this, write for $s < t$

$$\xi_t^x = \xi_s^x + \int_s^t a(\xi_s^x)dw + \int_s^t b(\xi_s^x)ds.$$

A similar calculation to above shows that the generator of ξ is the differential operator

$$A\phi(x) = \frac{1}{2}a^2(x)\phi''(x) + b(x)\phi'(x).$$

Thus the function $u(t,x) \equiv E[\phi(\xi_t^x)]$ solves

$$\frac{\partial u}{\partial t} = Au$$

with initial condition $u(0,\cdot) = \phi$.